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h-Almost conformal Ricci-Bourguignon soliton on generalized Sasakian space form with respect to quarter-symmetric metric connection

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Abstract. The purpose of the present paper is to discuss about a generalized Sasakian space form with quarter-symmetric metric connection satisfying h-almost conformal Ricci-Bourguignon soliton and h-almost conformal η-Ricci-Bourguignon soliton. Here, we have evolved the nature of h-almost conformal Ricci-Bourguignon soliton on a generalized Sasakian space form with quarter-symmetric metric connection when the potential vector field is to be considered as a conformal vector field, a torse-forming vector field or a torqued vector field. Then we have established that a generalized Sasakian space form with quarter-symmetric metric connection satisfying gradient h-conformal Ricci-Bourguignon soliton to turn out an Einstein manifold. Later, we have constructed Laplacian equation from h-almost conformal η-Ricci-Bourguignon soliton with quarter-symmetric metric connection when the potential vector field ξ is of gradient of a smooth function f. Finally we have examined the existence of an extended generalized ϕ-recurrent generalized Sasakian space form with quarter-symmetric metric connection endowing h-almost conformal η-Ricci-Bourguignon soliton.

Keywords: Ricci-Bourguignon soliton, h-almost conformal Ricci-Bourguignon soliton, h-almost conformal η-Ricci-Bourguignon soliton, generalized Sasakian space form, torse-forming vector field.

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1. **Introduction**

The concept of a soliton flows from the idea of a solitary wave that arises from a balance between nonlinear and dispersive effects which are associated with physical system. Solitons preserve their shapes and speeds while propagating freely, at constant velocity and retrieve it after collisions with other such waves. Solitons provide stable solutions of a wide class of weakly nonlinear dispersive partial differential equations describing physical systems. The concept of a Ricci flow was formulated in the early 1980s by R. Hamilton, who was developed by Eells and Sampson's work on harmonic map heat flow [12, 15]. Ricci flow is an evolution equation on a smooth manifold M with a Riemannian metric $g(t)$ defined as follows

$$\frac{\partial}{\partial t}g(t) = -2S.$$

Ricci soliton, which is a natural generalization of an Einstein manifold, is defined on a semi-Riemannian manifold (M, g) by

$$S + \frac{1}{2}\pounds_Y g = \mu g$$

where $\pounds_Y$ is the Lie derivative along the vector field Y, S is the Ricci tensor of (M, g) and μ is a real constant. If $Y = \nabla f$ for some function f on M, the Ricci soliton transforms into a gradient Ricci soliton. A soliton becomes shrinking, steady and expanding according as $\mu > 0$, $\mu = 0$ and $\mu < 0$ respectively.

In [4], N. Basu and A. Bhattacharyya constructed the notion of conformal Ricci soliton, defined as:

$$\pounds_V g + 2S + [2\mu - (p + \frac{2}{n})]g = 0, \tag{1.1}$$

where $\pounds_V$ is the Lie derivative along the vector field V, p defined as a scalar non-dynamical field(time dependent scalar field), μ is constant, n is the dimension of the manifold.

In 1979, the idea of the Ricci-Bourguignon flow (or RB flow) as a generalization of Ricci flow was developed by Jean-Pierre Bourguignon [6] using some unpublished work of Lichnerowicz and a paper of Aubin [2]. The Ricci-Bourguignon flow is an evolution equation for metrics on a Riemannian manifold given by

$$\frac{\partial}{\partial t}g(t) = -2(S - r\Lambda g), \tag{1.2}$$

where $\Lambda \in \mathbb{R}$ is a constant and r is the scalar curvature of the Riemannian metric g. It should be observed that the right hand side of the evolution equation (1.2) is of special interest for special values of Λ in particular [11].

1. $\Lambda = \frac{1}{2}$, the Einstein tensor $S - \frac{r}{2}g$ (Einstein soliton).
2. $\Lambda = \frac{1}{n}$, the traceless Ricci tensor $S - \frac{r}{n}g$.
3. $\Lambda = \frac{1}{2(n-1)}$, the Schouten tensor $S - \frac{r}{2(n-1)}g$ (Schouten soliton).
4. $\Lambda = 0$, the Einstein tensor S (Ricci soliton).

In [11], S. Dwivedi introduced the concept of Ricci-Bourguignon soliton which generalize Ricci solitons. In the paper, the author explained integral formulas for compact gradient Ricci-Bourguignon solitons and compact gradient Ricci-Bourguignon almost solitons.

A Riemannian manifold (M, g) is called a Ricci-Bourguignon soliton (or RB soliton) if there exists a smooth vector field V satisfying the following equation

$$S + \frac{1}{2}\pounds_V g = (\mu + r\Lambda)g, \tag{1.3}$$

for some real constant μ and the Lie derivative $\pounds_V g$.

The Ricci-Bourguignon soliton appears as a self-similar solution to Ricci-Bourguignon flow and often emerges as a limit of dilation of singularities in the Ricci-Bourguignon flow [7]. The Ricci-Bourguignon soliton is said to be shrinking, steady or expanding if μ is positive, zero or negative, respectively.

If the vector field V is the gradient of a smooth function f, then g is called a gradient Ricci-Bourguignon soliton and equation (1.3) becomes

$$\nabla\nabla f + S = (\mu + r\Lambda)g. \tag{1.4}$$

From the above identities (1.1) and (1.3), two new entities will be introduced now: (a) h-almost conformal Ricci-Bourguignon soliton which generalizes both conformal soliton and Ricci-Bourguignon soliton, and (b) h-almost conformal η-Ricci-Bourguignon soliton which generalizes both conformal soliton and η-Ricci-Bourguignon soliton.

An n-dimensional complete Riemannian or pseudo-Riemannian manifold (M, g) is said to be h-almost conformal Ricci-Bourguignon soliton, and denoted by (M^n, g, h, V, μ) if there exists a smooth vector field V satisfying the following equation

$$S + \frac{h}{2}\pounds_V g = (\mu - \frac{1}{2}(p + \frac{2}{n}) + r\Lambda)g, \tag{1.5}$$

for some smooth functions h and μ and the Lie derivative $\pounds_V g$.

The h-almost conformal Ricci-Bourguignon soliton is said to be shrinking, steady or expanding if μ is positive, zero or negative, respectively.

If the vector field V is the gradient of a smooth function f, then the soliton equation becomes

$$h\nabla\nabla f + S = (\mu - \frac{1}{2}(p + \frac{2}{n}) + r\Lambda)g, \tag{1.6}$$

and the soliton is called h-almost gradient conformal Ricci-Bourguignon soliton.

An n-dimensional complete Riemannian or pseudo-Riemannian manifold (M, g) is said to be h-almost conformal η-Ricci-Bourguignon soliton, and denoted by $(M^n, g, h, \xi, \mu, \beta)$ if there exists a smooth vector field V satisfying the following equation

$$S + \frac{h}{2}\pounds_V g = (\mu - \frac{1}{2}(p + \frac{2}{n}) + r\Lambda)g + \beta\eta \otimes \eta, \tag{1.7}$$

where h and μ are smooth functions, β is a real constant and η is a 1-form.

The h-almost conformal Ricci-Bourguignon soliton is said to be shrinking, steady or expanding if μ is positive, zero or negative, respectively.

If we consider the soliton vector field as a gradient of a smooth function f, then the soliton equation becomes

$$h\nabla\nabla f + S = (\mu - \frac{1}{2}(p + \frac{2}{n}) + r\Lambda)g + \beta\eta \otimes \eta, \tag{1.8}$$

and the soliton is called gradient h-almost conformal η-Ricci-Bourguignon soliton.

In [18], D. G. Prakasha, A. M. Ravindranatha, S. K. Chaubey, P. Veeresha and Y. J. Suh investigated some results of the h-almost Ricci solitons and h-almost gradient Ricci solitons on generalized Sasakian-space-forms. [10] S. K. Chaubey and Y. J. Suh worked on Ricci-Bourguignon solitons and the features of Fischer-Marsden conjecture within the framework of generalized Sasakian-space-forms with β-Kenmotsu structure. They also obtained generalized Sasakian-space-form with β-Kenmotsu structure satisfying the Fischer-Marsden equation to be a conformal gradient soliton. [20] A. Sardar and A. Sarkar distinguished Ricci-Yamabe solitons and gradient Ricci-Yamabe solitons on 3-dimensional generalized Sasakian space forms with quasi Sasakian metric. [17] S. Kumar, P. Kumar and B. Pal established some interesting results for solenoidal and concurrent vector fields on warped product space with almost Ricci-Bourguignon soliton.

Motivated by the above outcomes and explorations, we discover in this paper a generalized Sasakian space form with quarter-symmetric metric connection satisfying h-almost conformal Ricci-Bourguignon Soliton. The paper is arranged as below : In Sections 2 and 3, we present some characteristics of a generalized Sasakian space form and quarter-symmetric metric connection. In Section 4, the nature of the soliton on a generalized Sasakian space form with quarter-symmetric metric connection under some certain conditions have been discussed. Then we have investigated that a generalized Sasakian space form with quarter-symmetric metric connection satisfying gradient h-conformal Ricci-Bourguignon soliton to become an Einstein manifold. Later, we have analyzed Laplacian equation from h-almost conformal η-Ricci-Bourguignon soliton with quarter-symmetric metric connection when the potential vector field ξ is of gradient type. In Section 5, we inspected about a generalized Sasakian space form with quarter-symmetric metric connection with h-almost conformal η-Ricci-Bourguignon soliton and also have examined the existence of an extended generalized ϕ-recurrent generalized Sasakian space form with quarter-symmetric metric connection endowing h-almost conformal η-Ricci-Bourguignon soliton.

2. Generalized Sasakian Space Form

The sectional curvature of a Riemannian manifold (M, g) plays an important role in differential geometry. The curvature tensor of a Riemannian manifold with constant sectional curvature c is given by the following equation

$$R(X,Y)Z = c\{g(Y,Z)X - g(X,Z)Y\},$$

for all $X, Y, Z \in \chi(M)$. Then the Riemannian manifold with constant sectional curvature c is called a real-space form. A Riemannian manifold with constant sectional curvature c is said to be elliptic, hyperbolic or flat according as $c > 0$, $c < 0$ or $c = 0$.

Similarly, we can define constant holomorphic sectional curvature in the complex manifold. A Kähler manifold M^n is of constant holomorphic sectional curvature c if and only if

$$\begin{aligned} R(X,Y)Z = \frac{c}{4}\{&g(X,Z)Y - g(Y,Z)X + g(F(X),Z)F(Y) \\ &-g(F(Y),Z)F(X) + 2g(F(X),Y)F(Z)\}, \end{aligned}$$

for all $X, Y, Z \in \chi(M)$. Then the complex manifold with constant holomorphic sectional curvature c is called a complex-space form.

A $(2n+1)$-dimensional smooth manifold M with an almost contact structure (ϕ, ξ, η, g) satisfies the following conditions [5]

$$\phi^2 = -I + \eta \otimes \xi, \ \ \eta(\xi) = 1, \tag{2.1}$$

$$\phi\xi = 0, \ \ \eta \circ \phi = 0, \ \ \eta(X) = g(X, \xi), \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y). \tag{2.3}$$

P. Alegre, D. E. Blair and A. Carriazo introduced the concept of generalized Sasakian space form in [1]. An almost contact metric manifold M with an almost contact metric structure (ϕ, ξ, η, g) is called a generalized Sasakian space form if there exist three functions f_1, f_2, f_3 on M such that the curvature tensor R is given by

$$\begin{aligned} R(X,Y)Z = f_1\Big\{g(Y,Z)X - g(X,Z)Y\Big\} + f_2\Big\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X \\ +2g(X,\phi Y)\phi Z\Big\} + f_3\Big\{g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi + \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X\Big\}. \end{aligned} \tag{2.4}$$

for all vector fields X, Y, Z on M.
If $f_1 = \frac{c+3}{4}$, $f_2 = f_3 = \frac{c-1}{4}$ then M is a Sasakian space form.
If $f_1 = \frac{c-3}{4}$, $f_2 = f_3 = \frac{c+1}{4}$ then M is a Kenmotsu space form.
If $f_1 = f_2 = f_3 = \frac{c}{4}$ then M is a cosymplectic space form.

In a $(2n+1)$-dimensional generalized Sasakian space form $M^{(2n+1)}(f_1, f_2, f_3)$, we have the following relations [1]

$$\nabla_X \xi = (f_3 - f_1)\phi X, \tag{2.5}$$

$$(\nabla_X \phi)(Y) = (f_1 - f_3)[g(X,Y)\xi - \eta(Y)X], \tag{2.6}$$

$$(\nabla_X \eta)(Y) = g(\nabla_X \xi, Y) = (f_3 - f_1)g(\phi X, Y), \tag{2.7}$$

$$R(X,Y)\xi = (f_1 - f_3)[\eta(Y)X - \eta(X)Y], \tag{2.8}$$

$$\eta(R(X,Y)Z) = (f_1 - f_3)[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)], \tag{2.9}$$

$$QX = (2nf_1 + 3f_2 - f_3)X - (3f_2 + (2n-1)f_3)\eta(X)\xi, \tag{2.10}$$

$$S(X,Y) = (2nf_1 + 3f_2 - f_3)g(X,Y) - (3f_2 + (2n-1)f_3)\eta(X)\eta(Y), \tag{2.11}$$

$$Q\xi = 2n(f_1 - f_3)\xi, \tag{2.12}$$

$$S(X,\xi) = 2n(f_1 - f_3)\eta(X), \tag{2.13}$$

$$r = 2n(2n+1)f_1 + 6nf_2 - 4nf_3, \tag{2.14}$$

3. Quarter-Symmetric Metric connection

In 1924, the notion of a semi-symmetric linear connection on a differentiable manifold was introduced by Friedmann and Schouten [13]. The definition of metric connection with torsion on a Riemannian manifold, was provided by Hayden (1932) in [16]. In 1970, K. Yano [21] considered a semi-symmetric metric connection and studied some of its properties. Then in 1975, Golab [14] introduced the definition of a quarter-symmetric linear connection on a differentiable manifold, which is a generalization of semi-symmetric connection. A linear connection $\tilde{\nabla}$ on a Riemannian manifold (M, g) is said to be a quarter-symmetric connection if its torsion tensor T with respect to the connection $\tilde{\nabla}$ defined by

$$T(X,Y) = \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X,Y],$$

satisfies

$$T(X,Y) = \eta(Y)\phi X - \eta(X)\phi Y, \tag{3.1}$$

where η is a 1-form and ϕ is a $(1,1)$ tensor field.

A quarter-symmetric connection $\tilde{\nabla}$ is called a quarter-symmetric metric connection if $\tilde{\nabla} g = 0$. Let (M, g) be a contact metric manifold with the Levi-Civita connection ∇ and a linear connection $\tilde{\nabla}$ such that

$$\tilde{\nabla}_X Y = \nabla_X Y + G(X,Y), \tag{3.2}$$

where $G(X,Y)$ is a $(1,1)$-type tensor. For $\tilde{\nabla}$ to be a quarter-symmetric metric connection on M, we get

$$G(X,Y) = \frac{1}{2}[T(X,Y) + T'(X,Y) + T'(Y,X)], \tag{3.3}$$

where

$$g(T'(X,Y),Z) = g(T(Z,X),Y). \tag{3.4}$$

From the equations (3.1) we have

$$T'(X,Y) = g(\phi Y, X)\xi - \eta(X)\phi Y, \tag{3.5}$$

and using the equations (3.1) and (3.5) we derive

$$G(X,Y) = -\eta(X)\phi Y. \tag{3.6}$$

Hence a quarter-symmetric connection $\tilde{\nabla}$ on a generalized Sasakian space form can be written as

$$\tilde{\nabla}_X Y = \nabla_X Y - \eta(X)\phi Y. \tag{3.7}$$

Further, a relation between the curvature tensors R and $\tilde{R}$ of type (1,3) of the connections ∇ and $\tilde{\nabla}$, respectively is given by ,

$$\begin{aligned}
\tilde{R}(X,Y)Z &= R(X,Y)Z + 2(f_3 - f_1)g(X,\phi Y)\phi Z \\
&+ (f_1 - f_3)[\eta(X)g(Y,Z)\xi - \eta(Y)g(X,Z)\xi] \\
&+ (f_1 - f_3)\eta(Z)(\eta(Y)X - \eta(X)Y).
\end{aligned} \tag{3.8}$$

In the view of (3.8) we get

$$\tilde{S}(Y,Z) = S(Y,Z) - (f_1 - f_3)(g(Y,Z) - (2n+1)\eta(Y)\eta(Z)), \tag{3.9}$$

where $\tilde{S}$ and S are Ricci tensors with respect to quarter-symmetric metric connection $\tilde{\nabla}$ and the Levi-Civita connection ∇, respectively on M^{2n+1}. On a generalized Sasakian space form, the Ricci tensor of quarter-symmetric metric connection is symmetric.
From above, we have

$$\tilde{r} = r, \tag{3.10}$$

where $\tilde{r}$ and r are scalar curvature of M with respect to quarter-symmetric metric connection $\tilde{\nabla}$ and the Levi-Civita connection ∇, respectively.

4. h-Almost Conformal Ricci-Bourguignon Soliton on a Generalized Sasakian Space Form with Quarter-Symmetric Metric Connection

In this section, we will study h-almost conformal Ricci-Bourguignon soliton on a generalized Sasakian space form with quarter-symmetric metric connection and will establish some theorems about it.

Theorem 4.1. *If V is point-wise collinear with ξ on a generalized Sasakian space form $M^{2n+1}(f_1, f_2, f_3)$ with quarter-symmetric metric connection satisfying h-conformal Ricci-Bourguignon soliton (g, V, h, μ, Λ), h being a constant, then soliton is shrinking, steady or expanding if*

$$r\Lambda \lesseqqgtr 4n(f_1 - f_3) + \frac{1}{2}(p + \frac{1}{2}),$$

respectively.

Proof. First we consider that in a $(2n+1)$-dimensional generalized Sasakian space-form $M^{2n+1}(f_1, f_2, f_3)$ with quarter-symmetric metric connection satisfying h-almost conformal Ricci-Bourguignon soliton (g, V, h, μ, Λ) provides

$$\tilde{S}(X, Y) + \frac{h}{2}(\tilde{\mathcal{L}}_V g)(X, Y) = (\mu - \frac{1}{2}(p + \frac{1}{2}) + \tilde{r}\Lambda)g(X, Y). \tag{4.1}$$

Using (3.7), (3.9) and (3.10) we obtain

$$\begin{aligned} S(X, Y) \quad &- \quad (f_1 - f_3)(g(X, Y) - (2n+1)\eta(X)\eta(Y)) \\ &+ \quad \frac{h}{2}(g(\nabla_X V, Y) + g(\nabla_Y V, X) - \eta(X)g(Y, \phi V) - \eta(Y)g(X, \phi V)) \\ &= \quad (\mu - \frac{1}{2}(p + \frac{1}{2}) + r\Lambda)g(X, Y). \end{aligned} \tag{4.2}$$

Since, V is point-wise collinear with ξ i.e. $V = b\xi$, where b is a function, from the above it can be written as

$$\begin{aligned} S(X, Y) \quad &- \quad (f_1 - f_3)(g(X, Y) - (2n+1)\eta(X)\eta(Y)) \\ &+ \quad \frac{h}{2}(bg(\nabla_X \xi, Y) + bg(\nabla_Y \xi, X) + (Xb)\eta(Y) + (Yb)\eta(X)) \\ &= \quad (\mu - \frac{1}{2}(p + \frac{1}{2}) + r\Lambda)g(X, Y). \end{aligned} \tag{4.3}$$

Putting $Y = \xi$ in the equation (4.3) we get

$$\begin{aligned} 4n(f_1 - f_3)\eta(X) \quad &+ \quad \frac{h}{2}((Xb) + (\xi b)\eta(X)) \\ &= \quad (\mu - \frac{1}{2}(p + \frac{1}{2}) + r\Lambda)\eta(X). \end{aligned} \tag{4.4}$$

Replacing X by ξ in the above equation we deduce that

$$4n(f_1 - f_3) + h(\xi b) = \mu - \frac{1}{2}(p + \frac{1}{2}) + r\Lambda. \tag{4.5}$$

Putting the value of ξb in the equation (4.4) we derive

$$4n(f_1 - f_3)\eta(X) \quad + \quad h(Xb) = [(\mu - \frac{1}{2}(p + \frac{1}{2}) + r\Lambda)]\eta(X). \tag{4.6}$$

Taking exterior differentiation to (4.6) we get

$$[4n(f_1 - f_3) - (\mu - \frac{1}{2}(p + \frac{1}{2}) + r\Lambda)]d\eta = 0. \tag{4.7}$$

Since $d\eta \neq 0$ we get

$$\mu = 4n(f_1 - f_3) + \frac{1}{2}(p + \frac{1}{2}) - r\Lambda. \tag{4.8}$$

So, from the equation (4.8) we have

$$(db)(X) = 0 \Rightarrow b = \text{constant}. \tag{4.9}$$

Hence, the Ricci soliton is shrinking, expanding or steady if $r\Lambda < 4n(f_1 - f_3) + \frac{1}{2}(p + \frac{1}{2})$, $r\Lambda > 4n(f_1 - f_3) + \frac{1}{2}(p + \frac{1}{2})$, or $r\Lambda = 4n(f_1 - f_3) + \frac{1}{2}(p + \frac{1}{2})$ respectively. □

Definition 4.2. *A vector field V on a (pseudo) Riemannian manifold M is called a conformal vector field* [9] *if*

$$\pounds_\xi g = 2\rho g, \tag{4.10}$$

for a smooth function $\rho \in C^\infty(M)$.

On the other hand, a vector field τ on a semi-Riemannian manifold is called torse-forming if it satisfies

$$\nabla_X \tau = \varrho X + \upsilon(X)\tau, \tag{4.11}$$

for any $X \in \chi(M)$ with a smooth function ϱ and υ is a 1-form.

(i) The vector field is called concircular if υ vanishes identically.
(ii) The vector field is called concurrent if υ vanishes identically and $\varrho = 1$.
(iii) The vector field is called recurrent if $\varrho = 0$.
(iv) The vector field is called parallel if υ vanishes identically and $\varrho = 0$.
On a Riemannian or pseudo-Riemannian manifold, a nowhere zero vector field τ is called a torqued vector field if it satisfies [8]

$$\nabla_X \tau = \varrho X + \upsilon(X)\tau, \ \upsilon(\tau) = 0. \tag{4.12}$$

The function ϱ is called the torqued function and the 1-form υ is called the torqued form of τ.

Theorem 4.3. *Let $M^{2n+1}(f_1, f_2, f_3)$ a generalized Sasakian space form with the quarter-symmetric metric connection endowing the h-almost conformal Ricci-Bourguignon soliton (g, V, h, μ, Λ) where V is conformal vector field, then soliton is shrinking, steady or expanding if*

$$r\Lambda \lesseqqgtr 4n(f_1 - f_3) + h\rho + \frac{1}{2}(p + \frac{1}{2}),$$

respectively.

Proof. Let us consider that a (2n+1)-dimensional generalized Sasakian space-form $M^{2n+1}(f_1, f_2, f_3)$ with quarter-symmetric metric connection endows h-almost conformal Ricci-Bourguignon soliton (g, V, h, μ, Λ) then we can write

$$\tilde{S}(X,Y) + \frac{h}{2}(\tilde{\mathcal{L}}_V g)(X,Y) = (\mu - \frac{1}{2}(p + \frac{1}{2}) + \tilde{r}\Lambda)g(X,Y). \tag{4.13}$$

Using (3.7), (3.9) and (3.10) and (4.10) we obtain

$$\begin{aligned} S(X,Y) &- (f_1 - f_3)(g(X,Y) - (2n+1)\eta(X)\eta(Y)) \\ &+ \frac{h}{2}2\rho g(X,Y) - \frac{h}{2}\eta(X)g(Y,\phi V) - \frac{h}{2}\eta(Y)g(X,\phi V)) \\ &= (\mu - \frac{1}{2}(p + \frac{1}{2}) + r\Lambda)g(X,Y). \end{aligned} \tag{4.14}$$

Replacing Y with ξ, we have

$$4n(f_1 - f_3)\eta(X) + h\rho\eta(X) = (\mu - \frac{1}{2}(p + \frac{1}{2}) + r\Lambda + \beta)\eta(X). \tag{4.15}$$

Since $\eta(X) \neq 0$ we get

$$\mu = 4n(f_1 - f_3) + h\rho + \frac{1}{2}(p + \frac{1}{2}) - r\Lambda. \tag{4.16}$$

Hence this completes the proof. □

Theorem 4.4. *Let $M^{2n+1}(f_1, f_2, f_3)$ be a generalized Sasakian space form with the quarter-symmetric metric connection endowing h-almost conformal Ricci-Bourguignon soliton $(g, \tau, h, \mu, \Lambda)$ where τ is torse-forming vector field, then*
(i) the soliton is shrinking if $p > 2r(\Lambda - \frac{1}{2n+1}) - 2h(\varrho + \frac{1}{2n+1}\upsilon(\tau)) - \frac{1}{2}$,
(ii) the soliton is expanding if $p < 2r(\Lambda - \frac{1}{2n+1}) - 2h(\varrho + \frac{1}{2n+1}\upsilon(\tau)) - \frac{1}{2}$,
(iii) the soliton is steady if $p = 2r(\Lambda - \frac{1}{2n+1}) - 2h(\varrho + \frac{1}{2n+1}\upsilon(\tau)) - \frac{1}{2}$.

Proof. We assume that a (2n+1)-dimensional generalized Sasakian space-form $M^{2n+1}(f_1, f_2, f_3)$ with quarter-symmetric metric connection admits h-almost conformal Ricci-Bourguignon soliton $(g, \tau, h, \mu, \Lambda)$ then we obtain

$$\tilde{S}(X,Y) + \frac{h}{2}(\tilde{\mathcal{L}}_\tau g)(X,Y) = (\mu - \frac{1}{2}(p + \frac{1}{2}) + \tilde{r}\Lambda)g(X,Y). \tag{4.17}$$

Using the equation (4.11) the above identity can be noted as

$$\begin{aligned} \tilde{S}(X,Y) &+ \frac{h}{2}[g(\varrho X, Y) + \upsilon(X)g(\tau, Y) - \eta(X)g(\phi\tau, Y)) \\ &+ g(X, \varrho Y) + \upsilon(Y)g(\tau, X) - \eta(Y)g(\phi\tau, X))] \\ &= (\mu - \frac{1}{2}(p + \frac{1}{2}) + \tilde{r}\Lambda)g(X,Y). \end{aligned} \tag{4.18}$$

Using (3.9), (3.10) and (4.18) we obtain

$$\begin{aligned} S(X,Y) &- (f_1 - f_3)[g(X,Y) - (2n+1)\eta(X)\eta(Y)] \\ &+ \frac{h}{2}[g(\varrho X,Y) + \upsilon(X)g(\tau,Y) - \eta(X)g(\phi\tau,Y)) \\ &+ g(X,\varrho Y) + \upsilon(Y)g(\tau,X) - \eta(Y)g(\phi\tau,X))] \\ &= (\mu - \frac{1}{2}(p+\frac{1}{2}) + \tilde{r}\Lambda)g(X,Y). \end{aligned} \tag{4.19}$$

Contracting $X = Y = e_i$ $1 \leq i \leq (2n+1)$ we achieve

$$\mu = r(\frac{1}{2n+1} - \Lambda) + \frac{h}{2n+1}[\varrho(2n+1) + \upsilon(\tau)] + \frac{1}{2}(p+\frac{1}{2}). \tag{4.20}$$

Hence this proof is completed. □

Corollary 4.5. *Let $M^{2n+1}(f_1, f_2, f_3)$ be a generalized Sasakian space form with quarter-symmetric metric connection endowing h-almost conformal Ricci-Bourguignon soliton (g,τ,h,μ,Λ) where τ is concircular vector field, then the soliton is shrinking, expanding or steady if $p > 2r(\Lambda - \frac{1}{2n+1}) - 2h\varrho - \frac{1}{2}$, $p < 2r(\Lambda - \frac{1}{2n+1}) - 2h\varrho + -\frac{1}{2}$, or $p = 2r(\Lambda - \frac{1}{2n+1}) - 2h\varrho - \frac{1}{2}$, respectively.*

Corollary 4.6. *Let $M^{2n+1}(f_1, f_2, f_3)$ be a generalized Sasakian space form with quarter-symmetric metric connection endowing h-almost conformal Ricci-Bourguignon soliton (g,τ,h,μ,Λ) where τ is concurrent vector field, then the soliton is shrinking, expanding or steady if $p > 2r(\Lambda - \frac{1}{2n+1}) - 2h - \frac{1}{2}$, $p < 2r(\Lambda - \frac{1}{2n+1}) - 2h - \frac{1}{2}$, or $p = 2r(\Lambda - \frac{1}{2n+1}) - 2h - \frac{1}{2}$, respectively.*

Corollary 4.7. *Let $M^{2n+1}(f_1, f_2, f_3)$ be a generalized Sasakian space form with quarter-symmetric metric connection endowing h-almost conformal Ricci-Bourguignon soliton (g,τ,h,μ,Λ) where τ is recurrent vector field, then soliton is shrinking, expanding or steady if $p > 2r(\Lambda - \frac{1}{2n+1}) - \frac{1}{2n+1}2h\upsilon(\tau) - \frac{1}{2}$, $p < 2r(\Lambda - \frac{1}{2n+1}) - \frac{1}{2n+1}2h\upsilon(\tau) - \frac{1}{2}$, or $p = 2r(\Lambda - \frac{1}{2n+1}) - \frac{1}{2n+1}2h\upsilon(\tau) - \frac{1}{2}$, respectively.*

Corollary 4.8. *Let $M^{2n+1}(f_1, f_2, f_3)$ be a generalized Sasakian space form with quarter-symmetric metric connection endowing h-almost conformal Ricci-Bourguignon soliton (g,τ,h,μ,Λ) where τ is torqued vector field, then the soliton is shrinking, expanding or steady if $p > 2r(\Lambda - \frac{1}{2n+1}) - 2h - \frac{1}{2}$, $p < 2r(\Lambda - \frac{1}{2n+1}) - 2h - \frac{1}{2}$, or $p = 2r(\Lambda - \frac{1}{2n+1}) - 2h - \frac{1}{2}$, respectively.*

Corollary 4.9. *Let $M^{2n+1}(f_1, f_2, f_3)$ be a generalized Sasakian space form with quarter-symmetric metric connection endowing h-almost conformal Ricci-Bourguignon soliton (g,τ,h,μ,Λ) where τ is parallel vector field, then the soliton is shrinking, expanding or steady if $p > 2r(\Lambda - \frac{1}{2n+1}) - \frac{1}{2}$, $p < 2r(\Lambda - \frac{1}{2n+1}) - \frac{1}{2}$, or $p = 2r(\Lambda - \frac{1}{2n+1}) - \frac{1}{2}$, respectively.*

Theorem 4.10. *Let $M^{2n+1}(f_1, f_2, f_3)$ be a generalized Sasakian space form with quarter-symmetric metric connection satisfying gradient h-conformal Ricci-Bourguignon soliton where f_1, f_2, f_3 and h are constants then M becomes an Einstein manifold.*

Proof. Let $M^{2n+1}(f_1, f_2, f_3)$ a generalized Sasakian space form with quarter-symmetric metric connection satisfying gradient h-conformal Ricci-Bourguignon soliton. Then the equation (1.6) can be represented as

$$\tilde{\nabla}_X Df = \frac{1}{h}[(\mu - \frac{1}{2}(p + \frac{1}{2}) + \tilde{r}\Lambda)X - \tilde{Q}X]. \tag{4.21}$$

Covariant derivative of (4.21) along Y gives

$$\begin{aligned} \tilde{\nabla}_Y \tilde{\nabla}_X Df &= \frac{1}{h}[(\mu - \frac{1}{2}(p + \frac{1}{2}) + \tilde{r}\Lambda)\tilde{\nabla}_Y X \\ &- \tilde{\nabla}_Y \tilde{Q}X] \end{aligned} \tag{4.22}$$

Interchanging X and Y and making the view of the above equation in the relation $\tilde{R}(X,Y)Df = \tilde{\nabla}_X \tilde{\nabla}_Y Df - \tilde{\nabla}_Y \tilde{\nabla}_X Df - \tilde{\nabla}_[X,Y]Df$ we infer

$$h\tilde{R}(X,Y)Df = (\tilde{\nabla}_Y \tilde{Q})(X) - (\tilde{\nabla}_X \tilde{Q})(Y) + \Lambda(X\tilde{r})Y - \Lambda(Y\tilde{r})X \tag{4.23}$$

Also from the equation (3.9) we have

$$\tilde{Q}Y = ((2n-1)f_1 + 3f_2)Y - (3f_2 - (2n+1)f_1 + 4nf_3)\eta(Y)\xi \tag{4.24}$$

Using the equations (2.5), (2.7), (3.7) and (4.24) we can represent as

$$\begin{aligned} (\tilde{\nabla}_X \tilde{Q})Y &= -(3f_2 - (2n+1)f_1 + 4nf_3)(f_3 - f_1)g(\phi X, Y)\xi \\ &- ((2n-1)f_1 + 3f_2)\eta(X)\phi Y + 4n(f_3 - f_1)\eta(Y)\phi X. \end{aligned} \tag{4.25}$$

Also we have

$$\begin{aligned} (\tilde{\nabla}_Y \tilde{Q})X &= -(3f_2 - (2n+1)f_1 + 4nf_3)(f_3 - f_1)g(\phi Y, X)\xi \\ &- ((2n-1)f_1 + 3f_2)\eta(Y)\phi X + 4n(f_3 - f_1)\eta(X)\phi Y. \end{aligned} \tag{4.26}$$

Taking inner product with W in the equation (4.23) and putting $X = W = e_i$, $1 \le i \le (2n+1)$ and also using the equations (4.25) and (4.26) we obtain

$$h\tilde{S}(Y, Df) = -2n(Y\tilde{r}). \tag{4.27}$$

Also $\tilde{r} = r$ and f_1, f_2, f_3 being constants we get

$$h\tilde{S}(Y, Df) = 0. \tag{4.28}$$

From the equation (3.9) we derive

$$S(Y, Df) = (f_1 - f_3)[(Yf) - (2n+1)\eta(Y)(\xi f) \tag{4.29}$$

Replacing Y by ξ and using the equation (4.29) we can write

$$\xi f = 0 \tag{4.30}$$

Again in view of (4.23) it can be represented as

$$hg(\tilde{R}(X,Y)\xi, Df) = 2(f_1 - f_3)(3f_2 - (2n+1)f_1 + 4nf_3)g(\phi X, Y). \quad (4.31)$$

Using the equation (2.8), (3.8) and (4.31) we obtain

$$\begin{aligned} 2h(f_1 - f_3)(\eta(Y)(Xf) &- \eta(X)(Yf)) = 2(f_1 - f_3)(3f_2 \\ &- (2n+1)f_1 + 4nf_3)g(\phi X, Y). \end{aligned} \quad (4.32)$$

Taking $X = \xi$ the above equation becomes $Yf = 0$ which implies f is a constant. The equation (4.21) gives that M is an Einstein manifold. □

Definition 4.11. [19] *Let M be a $(2n+1)$-dimensional be a generalized Sasakian space form with quarter-symmetric metric connection. The pseudo-projective curvature tensor of M is given by*

$$\begin{aligned} P_\star(X,Y)Z &= a_0 R(X,Y)Z + a_1[S(Y,Z)X - S(X,Z)Y] \\ &- \frac{r}{4}(\frac{a_0}{3} + a_1)[g(Y,Z)X - g(X,Z)Y]. \end{aligned} \quad (4.33)$$

for all vector fields X, Y, Z and a_0, a_1 being constants.

Definition 4.12. *A generalized Sasakian space form is said to be ϕ-pseudo-projectivly flat with respect to quarter-symmetric metric connection if*

$$\phi^2(P_\star(\phi X, \phi Y)\phi Z)) = 0. \quad (4.34)$$

Theorem 4.13. *Let $M^{(2n+1)}$ be a $\phi-$ pseudo-projectively flat generalized Sasakian space form with respect to quarter-symmetric metric connection endowing h-almost conformal Ricci-Bourguignon soliton $(g, \xi, h, \mu, \Lambda)$, then soliton is shrinking steady or expanding if*

$$\begin{aligned} p &\gtreqless 2(f_3 - f_1) + 2\frac{r}{[}\Lambda + 2n + 1(\frac{a_0}{2na_1} + 1)] \\ &+ \frac{4a_0}{a_1(2n-1)}((n+1)f_2 + f_3 - f_1) + \frac{1}{2}(p + \frac{1}{2}) - \frac{1}{2}, \end{aligned}$$

respectively.

Proof. Let $M^{(2n+1)}$ be a ϕ-pseudo-projectivly flat generalized Sasakian space form with respect to quarter-symmetric metric connection. It is easy to see that $\phi^2(P_\star(\phi X, \phi Y)\phi Z)) = 0$ holds iff

$$g(\tilde{P}_\star(\phi X, \phi Y)\phi Z, \phi V)) = 0, \forall X, Y, Z, V \in \chi(M^{2n+1}). \quad (4.35)$$

Therefore, we obtain

$$
\begin{aligned}
g(\tilde{P}_{\star}(\phi X, \phi Y)\phi Z, \phi V)) &= a_0 g(\tilde{R}(\phi X, \phi Y)\phi Z, \phi V)) \\
&+ a_1(\tilde{S}(\phi Y, \phi Z)g(\phi X, \phi V) - \tilde{S}(\phi X, \phi Z)g(\phi Y, \phi V)) \\
&- \frac{1}{2n+1}(\frac{a_0}{2n} + a_1)\tilde{r}(g(\phi X, \phi V)g(\phi Y, \phi Z) \\
&- g(\phi Y, \phi V)g(\phi X, \phi Z)). \qquad (4.36)
\end{aligned}
$$

Using the equation (4.37) in the equation (4.36), we can write

$$
\begin{aligned}
a_0 g(\tilde{R}(\phi X, \phi Y)\phi Z, \phi V)) &= a_1(\tilde{S}(\phi X, \phi Z)g(\phi Y, \phi V) - \tilde{S}(\phi Y, \phi Z)g(\phi X, \phi V)) \\
&+ \frac{1}{2n+1}(\frac{a_0}{2n} + a_1)\tilde{r}(g(\phi X, \phi V)g(\phi Y, \phi Z) \\
&- g(\phi Y, \phi V)g(\phi X, \phi Z)). \qquad (4.37)
\end{aligned}
$$

Let $\{e_1, e_2, ...e_{2n}, e_{2n+1} = \xi\}$ be a local orthonormal basis of vector fields in M^{2n+1}. Then $\{\phi e_1, \phi e_2, ...\phi e_{2n}, \xi\}$ is also a local orthonormal basis of vector fields in M^{2n+1}. Putting $X = W = e_i$ in the equation (4.38) and taking summation over i, $1 \leq i \leq 2n+1$, we have

$$
\begin{aligned}
a_0 \sum_{i=1}^{2n+1} g(\tilde{R}(\phi e_i, \phi Y)\phi Z, \phi e_i)) &= a_1 \sum_{i=1}^{2n+1} (\tilde{S}(\phi e_i, \phi Z)g(\phi Y, \phi e_i) \\
&- \tilde{S}(\phi Y, \phi Z)g(\phi e_i, \phi e_i)) \\
&+ \sum_{i=1}^{2n+1} \frac{1}{2n+1}(\frac{a_0}{2n} + a_1)\tilde{r}(g(\phi e_i, \phi e_i)g(\phi Y, \phi Z) \\
&- g(\phi Y, \phi e_i)g(\phi e_i, \phi Z)). \qquad (4.38)
\end{aligned}
$$

Also,

$$\sum_{i=1}^{2n+1} g(\tilde{R}(\phi e_i, Y)Z, \phi e_i)) = 2((n+1)f_2 + f_3 - f_1)g(\phi Y, \phi Z), \qquad (4.39)$$

$$\sum_{i=1}^{2n+1} \tilde{S}(\phi e_i, Z)g(Y, \phi e_i) = S(\phi Y, \phi Z) - (f_1 - f_3)g(\phi Y, \phi Z), \qquad (4.40)$$

$$\sum_{i=1}^{2n+1} g(\phi e_i, \phi e_i) = 2n, \qquad (4.41)$$

$$\sum_{i=1}^{2n+1} g(\phi e_i, Z)g(Y, \phi e_i) = g(Y, Z), \qquad (4.42)$$

$$\sum_{i=1}^{2n+1} g(\phi e_i, e_i) = 0. \qquad (4.43)$$

Therefore, using the equations (4.39)-(4.44) in the equation (4.38) we derive

$$2a_0((n+1)f_2+f_3-f_1)g(\phi Y,\phi Z) = -(2n-1)a_1\Big(\tilde{S}(\phi Y,\phi Z) \\ -(f_1-f_3)g(\phi Y,\phi Z)\Big)+\frac{1}{2n+1}(\frac{a_0}{2n}+a_1)r(2n-1)g(\phi Y,\phi Z). \quad (4.44)$$

Then, from the equation (1.5) it can be represented as

$$S(\phi Y,\phi Z) = (\mu-\frac{1}{2}(p+\frac{1}{2})+r\Lambda)g(\phi Y,\phi Z). \quad (4.45)$$

Hence from the equations (4.45) and (4.46) we can write

$$\begin{aligned} 2a_0((n+1)f_2 &+ f_3-f_1)g(\phi Y,\phi Z) = -(2n-1)a_1((\mu-\frac{1}{2}(p+\frac{1}{2}) \\ &+ r\Lambda)g(\phi Y,\phi Z)-(f_1-f_3)g(\phi Y,\phi Z)) \\ &+ \frac{1}{2n+1}(\frac{a_0}{2n}+a_1)r(2n-1)g(\phi Y,\phi Z). \quad (4.46) \end{aligned}$$

Therefore, we get

$$\begin{aligned} \mu &= (f_1-f_3)+\frac{r}{2n+1}(\frac{a_0}{2na_1}+1) \\ &- \frac{2a_0}{a_1(2n-1)}((n+1)f_2+f_3-f_1)+\frac{1}{2}(p+\frac{1}{2})-r\Lambda. \quad (4.47) \end{aligned}$$

Hence this completes the proof. □

5. h-Almost Conformal η-Ricci-Bourguignon Soliton on a Generalized Sasakian Space Form with Quarter-Symmetric Metric Connection

Theorem 5.1. *Let $M^{2n+1}(f_1,f_2,f_3)$ a generalized Sasakian space form with quarter-symmetric metric connection satisfying h-almost conformal η-Ricci-Bourguignon soliton $(g,\xi,h,\mu,\Lambda,\beta)$, then μ and β are related by*

$$\mu+\beta = \frac{1}{2}(p+\frac{1}{2})+(2n(2n+1)\Lambda+2n)f_1+(6n\Lambda)f_2-(4n\Lambda+4n)f_3.$$

Proof. Let $M^{2n+1}(f_1,f_2,f_3)$ be a generalized Sasakian space form with the quarter-symmetric metric connection satisfying h- almost conformal η-Ricci-Bourguignon soliton $(g,\xi,h,\mu,\Lambda,\beta)$. Then we can write the equation (1.7) as

$$\tilde{S}(X,Y)+\frac{h}{2}(\tilde{\mathcal{L}}_V g)(X,Y) = (\mu-\frac{1}{2}(p+\frac{1}{2})+\tilde{r}\Lambda)g(X,Y)+\beta\eta(X)\eta(Y). \quad (5.1)$$

Using (3.7), (3.9) and (3.10) we have

$$\begin{aligned} S(X,Y) &- (f_1-f_3)(g(X,Y)-(2n+1)\eta(X)\eta(Y)) \\ &= (\mu-\frac{1}{2}(p+\frac{1}{2})+r\Lambda)g(X,Y)+\beta\eta(X)\eta(Y). \quad (5.2) \end{aligned}$$

Putting $Y = \xi$ in the equation (5.2) we obtain

$$4n(f_1 - f_3)\eta(X) = (\mu - \frac{1}{2}(p + \frac{1}{2}) + r\Lambda + \beta)\eta(X). \tag{5.3}$$

Since $\eta(X) \neq 0$ we get

$$\mu + \beta = \frac{1}{2}(p + \frac{1}{2}) + (2n(2n+1)\Lambda + 2n)f_1 + (6n\Lambda)f_2 - (4n\Lambda + 4n)f_3. \tag{5.4}$$

Hence, the theorem is proved. □

We have

$$\begin{aligned} S(X,Y) &- (f_1 - f_3)(g(X,Y) - (2n+1)\eta(X)\eta(Y)) \\ &+ \frac{h}{2}[g(\nabla_X \xi, Y) + g(\nabla_Y \xi, X)] \\ &= (\mu - \frac{1}{2}(p + \frac{1}{2}) + r\Lambda)g(X,Y) + \beta\eta(X)\eta(Y). \end{aligned} \tag{5.5}$$

Contracting $X = Y = e_i$ $1 \leq i \leq (2n+1)$ we obtain

$$r + \frac{h}{2}div\xi = (\mu - \frac{1}{2}(p + \frac{1}{2}) + r\Lambda)(2n+1) + \beta, \tag{5.6}$$

where $divV$is the divergence of the vector field ξ. If $\xi = gradf$ where $gradf$ is the gradient of the smooth function f, we get

$$\Delta f = \frac{2}{h}(\mu - \frac{1}{2}(p + \frac{1}{2}) + r\Lambda)(2n+1) + \frac{2\beta}{h} - \frac{2r}{h}. \tag{5.7}$$

Hence, we have the following.

Theorem 5.2. *Let $M^{2n+1}(f_1, f_2, f_3)$ be a generalized Sasakian space form with quarter-symmetric metric connection satisfying h-almost conformal η-Ricci-Bourguignon soliton $(g, \xi, h, \mu, \Lambda, \beta)$. If $\xi = gradf$, for a smooth function f then the Laplacian equation satisfied by f becomes*

$$\Delta f = \frac{2}{h}(\mu - \frac{1}{2}(p + \frac{1}{2}) + r\Lambda)(2n+1) + \frac{2\beta}{h} - \frac{2r}{h}.$$

Definition 5.3. *A Riemannian manifold (M^{2n+1}, g) is called $\phi-$generalized recurrent [3], if its curvature tensor R satisfies the condition*

$$\phi^2((\nabla_W R)(X,Y)Z) = A(W)R(X,Y)Z + B(W)[g(Y,Z)X - g(X,Z)Y],$$

where A and B are two 1-forms, B is non zero and these are defined by $g(W, \rho_1) = A(W)$ and $g(W, \rho_2) = B(W)$, $\forall W \in \chi(M)$. ρ_1 and ρ_2 being the vector fields associated to the 1-forms A and B respectively.

Theorem 5.4. *There does not exist an extended generalized ϕ-recurrent generalized Sasakian space form with quarter-symmetric metric connection satisfying h-almost conformal η-Ricci-Bourguignon soliton $(g, \xi, h, \mu, \Lambda, \beta)$.*

Proof. Let us assume an extended generalized ϕ- recurrent generalized Sasakian space form with quarter-symmetric metric connection endowing h-almost conformal η-Ricci-Bourguignon soliton $(g, \xi, h, \mu, \Lambda, \beta)$. Then we have

$$\phi^2((\tilde{\nabla}_W \tilde{R})(X,Y)Z) = A(W)\phi^2(\tilde{R}(X,Y)Z) + B(W)\phi^2([g(Y,Z)X - g(X,Z)Y]).$$

From above we get

$$\begin{aligned} &-(\tilde{\nabla}_W \tilde{R})(X,Y)Z + \eta((\tilde{\nabla}_W \tilde{R})(X,Y)Z)\xi \qquad (5.8)\\ &= A(W)[-\tilde{R}(X,Y)Z + \eta(\tilde{R}(X,Y)Z)\xi] + B(W)[-g(Y,Z)X + g(X,Z)Y\\ &\quad + g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi]. \end{aligned}$$

From the equations (2.5), (3.7), (2.7) and (3.8) and then taking inner product with U we have

$$\begin{aligned} -g((\nabla_W R)(X,Y)Z,U) &- 2(f_3 - f_1)g(X,(\nabla_W \phi)Y)g(\phi Z,U)\\ &- 2(f_3 - f_1)g(X,\phi Y)g((\nabla_W \phi)Z,U)\\ &- (f_1 - f_3)(\nabla_W \eta)(X)g(Y,Z)\eta(U)\\ &- (f_1 - f_3)\eta(X)g(\nabla_W \xi,U)g(Y,Z)\\ &+ (f_1 - f_3)\eta(Y)g(\nabla_W \xi,U)g(X,Z)\\ &+ (f_1 - f_3)(\nabla_W \eta)(X)g(Y,U)\eta(Z)\\ &- (f_1 - f_3)(\nabla_W \eta)(Y)g(X,U)\eta(Z)\\ &- (f_1 - f_3)(\nabla_W \eta)(Z)[\eta(Y)g(X,U) - \eta(X)g(Y,U)]\\ &+ \eta((\nabla_W R)(X,Y)Z)\eta(U)\\ &+ 2(f_3 - f_1)\eta((\nabla_W \phi)Z)\eta(U)g(X,\phi Y)\\ &+ (f_1 - f_3)(\nabla_W \eta)(X)g(Y,Z)\eta(U)\\ &+ (f_1 - f_3)\eta(\nabla_W \xi)g(Y,Z)\eta(X)\eta(U)\\ &- (f_1 - f_3)g(X,Z)\eta(Y)\eta(\nabla_W \xi)\eta(U)\\ &- (f_1 - f_3)(\nabla_W \eta)(X)\eta(Y)\eta(Z)\eta(U)\\ &+ (f_1 - f_3)(\nabla_W \eta)(Y)\eta(X)\eta(Z)\eta(U)\\ &= A(W)[-g(R(X,Y)Z,U) - 2(f_3 - f_1)g(X,\phi Y)g(\phi Z,U)\\ &- (f_1 - f_3)(g(Y,Z)\eta(X)\eta(U) - g(X,Z)\eta(Y)\eta(U))\\ &- (f_1 - f_3)\eta(Z)(g(X,U)\eta(Y) - \eta(X)g(Y,U))\\ &+ \eta(R(X,Y)Z)\eta(U)\\ &+ (f_1 - f_3)\eta(U)(\eta(X)g(Y,Z) - \eta(Y)g(X,Z))]\\ &+ B(W)[-g(Y,Z)g(X,U) + g(X,Z)g(Y,U)\\ &+ g(Y,Z)\eta(X)\eta(U) - g(X,Z)\eta(Y)\eta(U)]. \qquad (5.9) \end{aligned}$$

Let $\{e_1, e_2, ..., e_{2n+1} = \xi\}$ be an orthonormal basis for the tangent space of M^{2n+1} at a point $p \in M^{2n+1}$. Putting $X = U = e_i$ in (5.9) and taking summation over i, we get

$$\begin{aligned}
-(\nabla_W S)(Y,Z) &- 2(f_3 - f_1)g(\phi Z, (\nabla_W \phi)Y) \\
&+ (f_1 - f_3)\eta(Y)g(\nabla_W \xi, Z) \\
&+ (f_1 - f_3)(\nabla_W \eta)(e_i)g(Y, e_i)\eta(Z) \\
&- (2n+1)(f_1 - f_3)(\nabla_W \eta)(Y)\eta(Z) \\
&- 2n(f_1 - f_3)(\nabla_W \eta)(Z)\eta(Y) \\
&+ \eta((\nabla_W R)(e_i, Y)Z)\eta(e_i) \\
&= A(W)[-S(Y,Z) - 2(f_3 - f_1)g(\phi Y, \phi Z) \\
&- 2n(f_1 - f_3)(g(Y,Z)\eta(Y)\eta(Z) \\
&+ \eta(R(e_i, Y)Z)\eta(e_i)] \\
&+ B(W)[(1-2n)g(Y,Z) - \eta(Y)\eta(Z)]. \qquad (5.10)
\end{aligned}$$

Using the equations (2.5), (2.6) and (2.7) we obtain

$$\begin{aligned}
-(\nabla_W S)(Y,Z) &- 2(f_3 - f_1)^2\eta(Y)g(\phi Z, W) - 2(f_3 - f_1)^2\eta(Z)g(\phi Y, W) \\
&+ (f_1 - f_3)\eta(Z)g(\phi W, Y) + (2n+1)(f_1 - f_3)^2 g(\phi W, Y)\eta(Z) \\
&+ 2n(f_1 - f_3)^2 g(\phi W, Z)\eta(Y) \\
&+ \eta((\nabla_W R)(e_i, Y)Z)\eta(e_i) \\
&= A(W)[-S(Y,Z) - 2(f_3 - f_1)g(\phi Y, \phi Z) \\
&- 2n(f_1 - f_3)(g(Y,Z)\eta(Y)\eta(Z) + \eta(R(e_i, Y)Z)\eta(e_i)] \\
&- B(W)[(2n-1)g(Y,Z) + \eta(Y)\eta(Z)]. \qquad (5.11)
\end{aligned}$$

Also from the equation (5.2) we have

$$S(\xi, \xi) = \mu - \frac{1}{2}(p + \frac{1}{2}) + r\Lambda + \beta.$$

Again from (2.13) we can write

$$2n(f_1 - f_3) = \mu - \frac{1}{2}(p + \frac{1}{2}) + r\Lambda + \beta.$$

Putting $Y = Z = \xi$ in (5.11) and also using the equations (2.7), (2.9) and (2.13) it can be represented by

$$(2n+2)B(W) = 0.$$

As $n \neq -1$, Hence $B(W) = 0$ which is not possible. Hence our assumption is wrong. Therefore there does not exist an extended generalized ϕ-recurrent generalized Sasakian space form with quarter-symmetric metric connection satisfying h-almost conformal η-Ricci-Bourguignon soliton $(g, \xi, h, \mu, \Lambda, \beta)$. □

Remark 5.5. *The study of a h-almost conformal Ricci-Bourguignon soliton on semi Riemannian manifolds and Riemannian manifolds deals a significant role in the area of differential geometry and in special relativistic physics as well. Ricci-Bourguignon provides the most meaningful topic in modern physics. Here we have explored some important results of generalized Sasakian space form with h-almost conformal Ricci-Bourguignon soliton and h-almost conformal η-Ricci-Bourguignon soliton in terms of a quarter-symmetric metric connection. The the novel concept of h-almost conformal Ricci-Bourguignon soliton and h-almost conformal η-Ricci-Bourguignon soliton providesgeometric and physical applications with relativistic viscous fluid spacetime, admitting heat flux and stress, dark and dust fluid general relativistic spacetime, and radiation era in general relativistic spacetime. We will have more advantages to pursue the geometric properties. There are some questions that arises from our article to study in further research.*

(i) Which of the results of our paper are also true for trans-Sasakian manifolds, Co-Kähler manifold, or in para-contact geometry ?
(ii) Is the theorem 4.10 true for assuming h, f_1, f_2, f_3 as nonconstant functions ?
(iii) Does there exist an extended generalized ϕ-recurrent generalized Sasakian space form with h-almost conformal η-Ricci-Bourguignon soliton without quarter-symmetric metric connection ?

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