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Research Paper

SOME GRAPH PARAMETERS OF INDU-BALA PRODUCT OF GRAPHS

JOHN JOY MULLOOR[1], LIJU ALEX[2], G.INDULAL[3,*] AND DEENA C SCARIA[4]

[1]Department of Mathematics, St. Stephens College, Uzhavoor - 686634, Kerala, India, johnjoy.mulloor@gmail.com

[2]Department of Mathematics, Bishop Chulaparambil Memorial(BCM) College, Kottayam - 686001, Kerala, India, lijualex@bcmcollege.ac.in

[3,4]Department of Mathematics, St Aloysius College, Edathua-689573,Kerala, India,indulalgopal@gmail.com

[4]Department of Mathematics, Marthoma College, Thiruvalla, 689 103, Kerala, India,deenacelinescaria@gmail.com

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ABSTRACT

The Indu-Bala product of graphs G and H consists of two disjoint copies of the join of G and H such that there is an adjacency between the corresponding vertices in the two copies of H. A vertex subset S of a graph $G = (V, E)$ is said to be a geodetic set if every vertex in G is in some $u - v$ geodesic, where u and v are any two vertices in S. The minimum cardinality of such a set is the geodetic number of G. The vertex subset D of a graph G is said to be a dominating set if every vertex in G is either in D or adjacent to at least one vertex in D. The minimum cardinality of such a set is the domination number of G. In this work, the authors studied various geodetic and dominating extensions with respect to the Indu-Bala product of graphs. The A_α matrix associated with a graph is a convex linear combination of its adjacency matrix and degree diagonal matrix, offering deeper insights into the properties of both matrices. In this article the authors discuss the A_α spectrum of Indu-Bala product of graphs.

*Address correspondence to G. Indulal; Department of Mathematics, St.Aloysius College, Edathua, E-mail: indulalgopal@gmail.com.

1. Introduction

The graphs discussed in this study are simple, connected and non-directional. For the basic graph theoretic terminologies and notations we refer [3, 2]. For a graph $G = (V, E)$, the order of G is the cardinality of the vertex set V and size is the cardinality of the edge set E. The number of edges incident to a vertex v in G is its degree and is denoted as $deg(v)$. The maximum degree of a graph G is denoted as $\Delta(G)$ or Δ and the minimum degree of G is denoted as $\delta(G)$ or δ. The vertex with degree one is called the pendant vertex. A vertex v in a graph G is said to be a universal vertex if $deg(v) = |G| - 1$. The set of vertices that are adjacent to a vertex v is the neighbor of v and is denoted as $N(v)$. The set $N[v]$ is the union of $N(v)$ and $\{v\}$.

The distance between the two vertices u and v in a graph G is the number of edges in the shortest $u - v$ path or the $u - v$ geodesic. For any $u, v \in V(G)$, $I[u, v]$ denotes set of all vertices in some $u - v$ geodesic including u and v. For $S \subseteq V(G)$, $I[S]$ is the union of all vertices in the $u - v$ geodesic including u and v, where $u, v \in S$. The eccentricity of a vertex v in graph G is denoted as $e(v)$ and is defined as the distance between v and a vertex farthest from v in G. The maximum and the minimum eccentricities of vertices in G are the diameter and the radius of G, denoted by $diam(G)$ and $rad(G)$ respectively. A subgraph H of a graph G is said to be an induced subgraph [3] of G if each edge in G which ends in $V(H)$ also belongs to $E(H)$ and it is denoted as $< H >$.

Definition 1.1. [24] *A vertex v in a graph G is said to be an extreme vertex if $< N(v) >$ is complete. The number of extreme vertices in a graph G is its extreme order and is denoted as $ex(G)$.*

Definition 1.2. [14, 7] *The join of graphs G and H is denoted as $G + H$ and it consists of the vertex set*

$$V(G + H) = V(G) \cup V(G)$$

and the edge set

$$E(G + H) = E(G) \cup E(H) \cup \{xy : x \in V(G), y \in V(H)\}.$$

Definition 1.3. [20] *The Indu-Bala product of two graphs G and H is denoted as $G\blacktriangledown H$ and it consists of two disjoint copies of $G + H$ such that there is an adjacency between the corresponding vertices in the two copies of H.*

The Figure 1 represents the Indu-Bala product of paths P_3 and P_4.

Remark 1.1. *Generally, $G\blacktriangledown H \ncong H\blacktriangledown G$.*

Remark 1.2. *For any graphs G and H, $diam(G\blacktriangledown H) = 3$.*

Definition 1.4. [18, 19] *A vertex subset D of a graph G is a dominating set if every vertex in G is either in D or adjacent to at least one vertex in D. The minimum cardinality of such a set is the domination number of the graph, denoted by $\gamma(G)$.*

Definition 1.5. [17, 4, 5, 24] *A vertex subset S of a graph G is said to be a geodetic set, if $I[S] = V(G)$. The minimum cardinality of such a set is the geodetic number of G, denoted as $g(G)$.*

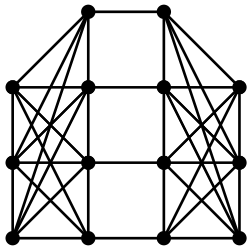

Figure 1. $P_3 \blacktriangledown P_4$

Definition 1.6. [16] *A vertex subset S of a graph G is said to be a geodetic dominating set if S is both a geodetic and dominating set of G. The minimum cardinality of such a set is the geodetic domination number, denoted as $\gamma_g(G)$.*

Definition 1.7. [1] *A geodetic set S of a graph $G = (V, E)$ is said to be a restrained geodetic set if $< V - S >$ has no isolated vertex. The minimum cardinality of such a set is the restrained geodetic number, which is denoted as $g_r(G)$.*

Definition 1.8. [12] *A dominating set D of a graph $G = (V, E)$ is said to be a restrained dominating set if $< V - D >$ has no isolated vertex. The minimum cardinality of such a set is the restrained domination number, denoted as $\gamma_r(G)$.*

Definition 1.9. [8] *A dominating set D of a graph $G = (V, E)$ is said to be a total dominating set if $< D >$ has no isolated vertex. The minimum cardinality of such a set is the total domination number, denoted as $\gamma_t(G)$.*

Definition 1.10. [21] *A dominating set D of a graph $G = (V, E)$ is said to be a total restrained dominating set if both $< V - D >$ and $< D >$ has no isolated vertex. The minimum cardinality of such a set is the total domination number, denoted as $\gamma_{tr}(G)$.*

Definition 1.11. [13] *A dominating set D of a graph $G = (V, E)$ is said to be a connected dominating set if $< D >$ is connected. The minimum cardinality of such a set is the connected domination number, denoted as $\gamma_c(G)$.*

Definition 1.12. [9, 6] *For a graph $G = (V, E)$, the Roman dominating function on G is a function $f : V(G) \to \{0, 1, 2\}$ with the condition that every vertex u with $f(u) = 0$ must adjacent to at least one vertex v such that $f(v) = 2$.*

Definition 1.13. [9, 6] *The weight of a Roman dominating function of a graph $G = (V, E)$ is defined as $\sum_{u \in V(G)} f(u)$. The minimum weight of a Roman dominating function of a graph G is the Roman domination number of G, denoted as $\gamma_R(G)$.*

Definition 1.14. [9] *A graph G is said to be a Roman graph, if $\gamma_R(G) = 2\gamma(G)$.*

Definition 1.15. [22] *Let $A(G)$ and $D(G)$ be the adjacency matrix and the diagonal matrix of vertex degrees of a graph $G = (V, E)$. Then the A_α spectrum of G is defined as the collection of eigenvalues of the matrix $A_\alpha(G)$ where,*

$$A_\alpha(G) = \alpha D(G) + (1 - \alpha)A(G)$$

for any $\alpha \in [0, 1]$.

Theorem 1.1. [24] *The pendant and extreme vertices set in a graph G is always a subset of any geodetic set of G.*

Theorem 1.2. [24, 5, 16, 1] *Let G be a graph of order $n > 1$. Then*

(a). $2 \leq g(G) \leq n$,
(b). $g(G) \leq n - diam(G) + 1$,
(c). $2 \leq g(G) \leq g_r(G) \leq n$,
(d). $2 \leq \min\{g(G), \gamma(G)\} \leq \gamma_g(G) \leq n$

Some extensions of dominating sets with respect to Indu-Bala products are discussed in [2, 25]. For some recent works see [28, 29] and the references cited there in. In the following section, we are discussing the extensions of the domination concept such as restrained domination, total domination, connected domination and Roman domination on Indu-Bala product of graphs.

2. Dominating Sets and its Extensions on Indu-Bala Product Graphs

Theorem 2.1. *Let G and H be any two connected graphs of order m and n respectively. Then $\gamma(G\blacktriangledown H) = \gamma(H\blacktriangledown G) = 2$ or 4. In particular,*

(a). $\gamma(G\blacktriangledown H) = \gamma(H\blacktriangledown G) = 2$ *if and only if* $\Delta(G) = m - 1$ *or* $\Delta(H) = n - 1$ *or* $\Delta(G) = m - 1$ *and* $\Delta(H) = n - 1$,
(b). $\gamma(G\blacktriangledown H) = \gamma(H\blacktriangledown G) = 4$ *if and only if* $\Delta(G) \neq m - 1$ *and* $\Delta(H) \neq n - 1$.

Proof. Without loss of generality let us consider $\Delta(G) = m - 1$. Then G has at least one universal vertex say u. Then, in both $G\blacktriangledown H$ and $H\blacktriangledown G$, u is adjacent to all other vertices of one copy of $G + H$. If $u^{'}$ is the universal vertex in the another copy of G, then $u^{'}$ is adjacent all other vertices of another copy of $G + H$ in $G\blacktriangledown H$ and $H\blacktriangledown G$. Also, there is no universal vertex in $G\blacktriangledown H$ and $H\blacktriangledown G$. Therefore, the set $D = \{u, u^{'}\}$ form a minimum dominating set of $G\blacktriangledown H$ and $H\blacktriangledown G$. Hence $\gamma(G\blacktriangledown H) = \gamma(H\blacktriangledown G) = 2$.

Conversely, let $\gamma(G\blacktriangledown H) = \gamma(H\blacktriangledown G) = 2$. Then, clearly, the vertices in the minimum dominating set of these graphs should belong to the two copies of $G + H$ and each of these vertices must be adjacent to all other vertices in that copy of $G + H$. This is possible only when $\Delta(G) = m - 1$ or $\Delta(H) = n - 1$ or $\Delta(G) = m - 1$ and $\Delta(H) = n - 1$.

Now consider $\Delta(G) \neq m - 1$ and $\Delta(H) \neq n - 1$. Then $\gamma(G\blacktriangledown H) \neq 2 \neq \gamma(H\blacktriangledown G)$. Then in $G\blacktriangledown H$, each vertex in G is dominated by a vertex in H, and each vertex in H is dominated by a vertex in G. Similarly for the other copies of G and H. Therefore, the minimum dominating set of $G\blacktriangledown H$ and $H\blacktriangledown G$ contain four vertices. The converse part is true from the above result. □

Corollary 2.1.1. *Let G and H be any two connected graphs. If G or H has at least one universal vertex, then $\gamma_r(G\blacktriangledown H) = 2$.*

Proof. Let u and $u^{'}$ be the universal vertices in the two copies of H. Then, from Theorem 2.1, $\gamma(G\blacktriangledown H) = 2$. The set $D = \{u, u^{'}\}$ form the minimum dominating set of $G\blacktriangledown H$. Also, $< V(G\blacktriangledown H) - S >$ is connected. Hence $\gamma_r(G\blacktriangledown H) = 2$. Similarly, we can prove if G has at least one universal vertex. □

Theorem 2.2. *Let G and H be non-trivial connected graphs. Then, $\gamma_c(G\blacktriangledown H) = 2$ if and only if H has at least one universal vertex, otherwise it is 4.*

Proof. Let H have at least one universal vertex say u and $u^{'}$ be the universal vertex in the another copy of H. Let $S = \{u, u^{'}\}$. Then S is a minimum dominating set of $G\blacktriangledown H$ and $< S >$ is P_2. Therefore, $\gamma_c(G\blacktriangledown H) = 2$.

Conversely let $\gamma_c(G\blacktriangledown H) = 2$ and $S = \{u, v\}$ be the minimum connected dominating set of $G\blacktriangledown H$. Clearly $< S >$ is P_2. If u and v are in the two copies of G, then $< S >$ is disconnected, which is impossible. If $u \in V(G)$ and $v \in V(H)$, then in $G\blacktriangledown H$ either S is not a dominating set or $< S >$ is disconnected. Therefore, u and v are in the two copies of H. Clearly, in $G\blacktriangledown H$, u is adjacent to all vertices of one copy of G and v is adjacent to all vertices of another copy of G. Since S is a dominating set, u must be adjacent to all other vertices in H. Similarly, v must be adjacent to all other vertices in another copy of H. Then u and v are universal vertices in H and its copy.

Consider the case in which H has no universal vertex and let S be the minimum connected dominating set of $G\blacktriangledown H$. Since $< S >$ is connected, S should contain at least one vertex from each copy of H. Since H has no universal vertex, in order to satisfy the domination condition we need to choose one vertex from each copy of G. Therefore, $\gamma_c(G\blacktriangledown H) = 4$. □

Corollary 2.2.1. *Let G and H be non-trivial connected graphs. Then, $\gamma_t(G\blacktriangledown H) = 2$ if and only if H has at least one universal vertex, otherwise it is 4.*

Proof. Let H have at least one universal vertex say u. Then the other copy of H also has universal vertex $u^{'}$. Then the set $S = \{u, u^{'}\}$ is a minimum dominating set of $G\blacktriangledown H$ and $< S >$ is P_2. Therefore, S is also a minimum total dominating set of $G\blacktriangledown H$ and hence $\gamma_t(G\blacktriangledown H) = 2$. Then the rest of the proof is similar to the proof of Theorem 2.2. □

Theorem 2.3. *Let G and H be non-trivial connected graphs. Then, $\gamma_{tr}(G\blacktriangledown H) = 2$ if and only if H has at least one universal vertex otherwise it is 4.*

Proof. Let u and $u^{'}$ be the universal vertices in H and its copy. Then from Corollary 2.2.1, the set $S = \{u, u^{'}\}$ is a minimum total dominating set of $G\blacktriangledown H$. By Corollary 2.1.1 S satisfies restrained condition also. Hence $\gamma_{tr}(G\blacktriangledown H) = 2$. Let H and its copy have no universal vertex, but G and its copy have a universal vertex. Then these vertices cannot form a minimum total restrained dominating set of $G\blacktriangledown H$ since the total domination condition does not hold. Then definitely $\gamma_{tr}(G\blacktriangledown H) > 2$. Therefore, if $\gamma_{tr}(G\blacktriangledown H) = 2$, then H and its copy must have universal vertex. The rest of the proof is similar to the proof of Theorem 2.2. □

Theorem 2.4. *Let G and H be non-trivial connected graphs. If G or H has at least one universal vertex, then $\gamma_R(G\blacktriangledown H) = 4$.*

Proof. Without loss of generality, let us consider G has at least one universal vertex. Let u and $u^{'}$ be the universal vertices in the two copies of G. Then, in $G\blacktriangledown H$ every vertex of one copy of $G + H$ is adjacent to u and every vertex of another copy of $G + H$ is adjacent to $u^{'}$. Therefore, if we are assigning the Roman dominating function f with $f(u) = 2 = f(u^{'})$, then the weight of all other vertices can be given as 0. Hence $\sum_{v\in V(G\blacktriangledown H)} = 4$. Therefore,

$\gamma_R(G\blacktriangledown H) \leq 4$.

Claim: $\gamma_R(G\blacktriangledown H) = 4$.
From Definition 1.3, $G\blacktriangledown H$ consists of two copies of $G + H$. If we are assigning the weight 0 to any arbitrary vertex of G or H, then there will be at least two vertices with weight 2. Hence $\sum_{v\in V(G\blacktriangledown H)} \geq 4$. If we are assigning the weight 1 to all the vertices, then $\sum_{v\in V(G\blacktriangledown H)} = 2(|G| + |H|) \geq 4$. So we need not consider that case. Clearly, we cannot assign exactly 4 or 3 or 2 or 1 vertices with weight 1 and rest all vertices with weight 0. Now consider the cases in which exactly one vertex with weight 2, one or two or three vertices with weight 1 and the rest with weight 0. But these cases are not possible. Since u is adjacent to all other vertices in one copy of $G + H$ and u' is adjacent to all vertices of another copy, $G\blacktriangledown H$ should have at least two vertices with weight 2. Then the only possibility of getting a minimum value for $\sum_{v\in V(G\blacktriangledown H)}$ is as mentioned in the previous paragraph. Hence $\gamma_R(G\blacktriangledown H) = 4$.

□

Corollary 2.4.1. *Let G and H be non-trivial connected graphs with orders m and n respectively. If $\Delta(G) = m - 1$ or $\Delta(H) = n - 1$, then $G\blacktriangledown H$ is a Roman graph.*

Proof. The result is true by Theorem 2.1 and Theorem 2.4. □

Corollary 2.4.2. *For every even number $n > 3$, there exists a Roman graph of order n.*

Proof. Consider any two graphs G and H with order m_1 and m_2 respectively, where $m_1 + m_2 = n$. Also, $\Delta(G) = m_1 - 1$. Then the graph $G\blacktriangledown H$ is of order n and by Theorem 2.1 and Theorem 2.4 $G\blacktriangledown H$ is a Roman graph. □

In the following section, we are discussing the geodetic concept and its various extensions in Indu-Bala product graphs.

3. Geodetic Sets and its Extensions on Indu-Bala Product Graphs

Proposition 3.1. *Let $G = G_1\blacktriangledown G_2$. Then $g(G) \leq |G| - 2$.*

Proof. Since $diam(G) = 3$, the result is true by the Theorem 1.2. □

Theorem 3.1. *Let $G \neq K_n$ be a graph of order $n > 2$ and H be any graph of order $m > 1$. Then, $g(H\blacktriangledown G) \leq 6$, $\gamma_g(H\blacktriangledown G) \leq 6$ and $g_r(H\blacktriangledown G) \leq 6$*

Proof. Let u_i, u_j and u_i', u_j' be the pair of non-adjacent vertices in the two copies of G. Let v_r, v_s' be the vertices in two copies of H. Now consider the set $S = \{u_i, u_j, u_i', u_j', v_r, v_s'\}$. Then S is a dominating set of $H\blacktriangledown G$ and $I[S] = V(H\blacktriangledown G)$. But $< V(H\blacktriangledown G) - S >$ is connected. Therefore, S is a geodetic, geodetic dominating and restrained geodetic set of $H\blacktriangledown G$, hence the result. □

Theorem 3.2. *For the graphs K_m and K_n,*

$$g(K_m\blacktriangledown K_n) = g_r(K_m\blacktriangledown K_n) = \gamma_g(K_m\blacktriangledown K_n) = 2m.$$

Proof. Let S be the set of vertices in the two copies of K_m. Then, these set of vertices are extreme in $K_m\blacktriangledown K_n$ and S is a dominating set of $K_m\blacktriangledown K_n$. Also, $I[S] = V(K_m\blacktriangledown K_n)$ and

$< V(K_m \blacktriangledown K_n) - S >$ is connected. Therefore, S is both a geodetic and restrained geodetic set of $K_m \blacktriangledown K_n$ with minimum cardinality and hence the result. □

Theorem 3.3. *Let G be any non-trivial graph of order n. Then, $g(H \blacktriangledown G) = \gamma_g(H \blacktriangledown G) = g_r(H \blacktriangledown G) = 2$, if and only if $H \cong K_1$.*

Proof. Let $g(H \blacktriangledown G) = \gamma_g(H \blacktriangledown G) = g_r(H \blacktriangledown G) = 2$ and $S = \{u, v\}$ be its minimum geodetic dominating set. Since S is a dominating set, every other vertices in $H \blacktriangledown G$ are adjacent to either u or v. Since $I[S] = 2$, $V(H \blacktriangledown G) - S$ is in $u - v$ geodesic. Also, $< V(H \blacktriangledown G) - S >$ is connected. Then, by the definition of Indu-Bala product of two graphs, H must be K_1. The converse part is true, since the vertices in two copies of K_1 form a geodetic and dominating set of $K_1 \blacktriangledown G$ and also satisfies the restrained property. □

Theorem 3.4. *Let G be any non-trivial graph of order n. Then,*

(a). $g(G \blacktriangledown \overline{K}_2) = \gamma_g(G \blacktriangledown \overline{K}_2) = g_r(G \blacktriangledown \overline{K}_2) = 4$,
(b). $g(\overline{K}_2 \blacktriangledown G) = \gamma_g(\overline{K}_2 \blacktriangledown G) = g_r(\overline{K}_2 \blacktriangledown G) = 4$.

Proof. Let $S = \{u_1, u_2, u_1^{'}, u_2^{'}\}$ be the vertices in the two copies of $\overline{K}_2$. Then S is a dominating set of both $G \blacktriangledown \overline{K}_2$ and $\overline{K}_2 \blacktriangledown G$. But, $< V(G \blacktriangledown \overline{K}_2) - S >$ consists of two disconnected copies of G and $< V(\overline{K}_2 \blacktriangledown G) - S >$ is connected. In $G \blacktriangledown \overline{K}_2$, vertices in each copy of G is in the geodesic between the vertices in $\overline{K}_2$, which is join with G. Also $I[S] = V(\overline{K}_2 \blacktriangledown G)$. Hence the results. □

Theorem 3.5. *Let $G = (K_n - e) \blacktriangledown H$, where H is a graph of order m and $n > 3$. Then,$g(G) = \gamma_g(G) = g_r(G) = 4$.*

Proof. Let u_i, u_j and $u_i^{'}, u_j^{'}$ be the non-adjacent vertices in the two copies of $K_n - e$. Then consider the set $S = \{u_i, u_j, u_i^{'}, u_j^{'}\}$. Clearly, S is a dominating set of G and each vertex in S are extreme vertex of G. But $I[S] = V(G)$ and $< V(G) - S >$ is $G = K_{n-2} \blacktriangledown H$. Therefore, $g(G) = \gamma_g(G) = g_r(G) = 4$. □

Theorem 3.6. *For $m > 3$, let $G = H \blacktriangledown (K_m - e)$, where $H \neq K_n$ is a graph of order n. Then, $g(G) = \gamma_g(G) = g_r(G) = 4$.*

Proof. Let H_1 and H_2 be the two copies of $H + K_m - e$. Then the non-adjacent vertices in $K_m - e$ form a minimum geodetic set for H_1 and H_2. Hence $g(H_1) = g(H_2) = 2$. Then these non-adjacent vertices form a minimum geodetic set S of G. Also, S is a dominating set and $< V(G) - S >$ is $H \blacktriangledown K_{m-2}$. Therefore, the result is true by applying Theorem 1.2. □

Theorem 3.7. *For $m \geq n \geq 3$, let $G = \overline{K}_m \blacktriangledown \overline{K}_n$. Then,*

$$g(G) = \gamma_g(G) = g_r(G) = 6.$$

Proof. From Theorem 3.4, all vertices in the two copies of $\overline{K}_m$ are in the geodesic between any two vertices from each copy of $\overline{K}_n$. Let S be the set of these vertices from the two copies of $\overline{K}_n$. Clearly S is not a geodetic set of G. Let u_i be a vertex in one copy of $\overline{K}_m$ and $u_i^{'}$ vertex in the another copy of $\overline{K}_m$. Now consider the set $S_1 = S \cup \{u_i, u_i^{'}\}$. Since all vertices in the two copies of $\overline{K}_n$ are in $u_i - u_i^{'}$ geodesic, $I[S_1] = V(G)$. Therefore, S_1 is a minimum geodetic set of G. Also, S_1 is a dominating set of G and $< V(G) - S_1 >$ is connected. Hence the result. □

Theorem 3.8. *Let H be a connected graph of order n and $G_1 = \overline{K}_m + H$, where $2 \leq m < n$. If $G = G_1 \blacktriangledown K_p$, then $g(G) = 2m$ or 8.*

Proof. e prove this theorem by considering the following only two cases.
Case 1: $H = K_n$
Then in G_1, each vertices in $\overline{K}_m$ are extreme, which form the minimum geodetic set of G_1 and hence $g(G_1) = m$. Let S be the set of vertices in the two copies of $\overline{K}_m$ in G. Then each $x \in V(G) - S$ is in the geodesic between any two vertices in S and $I[S] = V(G)$. Clearly, the vertices in the two copies of K_p are not in the minimum geodetic set of G and S is the minimum geodetic set. Hence $g(G) = 2m$.

Case 1: $H \neq K_n$
Let x, y be any two non-adjacent vertices in H. Then, in G_1 any vertex in $\overline{K}_m$ is in the $x - y$ geodesic. Let u_j, u_k be any two vertices in $\overline{K}_m$. Then, in G_1 any vertex in H is in $u_j - u_k$ geodesic. Therefore, the set $S = \{x, y, u_j, u_k\}$ form the minimum geodetic set of G_1 and hence $g(G_1) = 4$. Now consider the vertex set $S' = \{x, y, u_j, u_k, x', y', u_j', u_k'\}$ in G, which belongs to the two copies of G_1 in G. Clearly, in G, $d(x, y) = 2 = d(x', y')$. Also, any vertex in the two copies of K_p lies in the geodesic between any two vertices in S' which are on the two copies of G_1. Since G consists of two copies of $\overline{K}_m + H$, two non-adjacent vertices from each of these $\overline{K}_m$ and H are required to form a minimum geodetic set of G. Therefore, S' is a minimum geodetic set of G and hence $g(G) = 8$. □

4. A_α Spectrum of Indu-Bala Product Graphs

The adjacency and Laplacian spectra of Indu-Bala product of graphs were discussed in [23]. In this section, we are discussing A_α Spectrum with respect to the operation Indu-Bala product.

Definition 4.1. [10] *For an $n \times n$ matrix M, the $M-$coronal is defined as $\Gamma_M(x) = 1_n^T(xI_n - M)^{-1}1_n$, where 1_n is the column vector of size n with all entries one.*

Definition 4.2. [15] *The graphs G and H are considered cospectral if G and H have the same spectrum.*

Lemma 4.1. [27] *Let A, B, C and D be matrices with D is invertible. Then*

$$det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = det(D) - det(A - BD^{-1}C),$$

where $A - BD^{-1}C$ is called the Schur complement of D.

Lemma 4.2. [11] *Let $A = \begin{pmatrix} B & C \\ C & B \end{pmatrix}$ be a 2×2 block symmetric matrix. Then the eigenvalues of A are the eigenvalues of $B + C$ and the eigenvalues of $B - C$.*

Lemma 4.3. [10] *Let the row sum of each row in an $n \times n$ matrix A is t. Then $\Gamma_A(x) = \frac{n}{x-t}$.*

Theorem 4.1. *Let G and H be two graphs. Then*

$$\begin{aligned}P_{A_\alpha(G\blacktriangledown H)}(x) = {} & P_{A_\alpha(H)}(x-\alpha(n+1))P_{A_\alpha(G)}(x-\alpha m)\\ & \big((1-(1-\alpha)^2\Gamma_{A_\alpha(G)}(x-\alpha(n+1))\Gamma_{A_\alpha(H)}(x-\alpha m)\big)\, P_{A_\alpha(H)}(x+\alpha(n+1))\\ & P_{A_\alpha(G)}(x-\alpha m)\,\big((1-(1-\alpha)^2\Gamma_{A_\alpha(G)}(x+\alpha(n+1))\Gamma_{A_\alpha(H)}(x-\alpha m)\big)\end{aligned}$$

Proof. Let G and H be any two connected graphs of order m and n respectively and $\alpha \in [0,1]$. Then the matrix $A_\alpha(G\blacktriangledown H)$ is of order $2(m+n)\times 2(m+n)$ which is defined as follows.

$$A_\alpha(G\blacktriangledown H) = \begin{pmatrix} A_{n\times n} & B_{n\times m} & 0_{n\times m} & 0_{n\times n}\\ B^T_{m\times n} & C_{m\times m} & D_{m\times m} & 0_{m\times n}\\ 0_{m\times n} & D_{m\times m} & C_{m\times m} & B^T_{m\times n}\\ 0_{n\times n} & 0_{n\times n} & B_{n\times m} & A_{n\times n}\end{pmatrix}$$

where, $A = A_\alpha(G) + \alpha\ mI_n$, $B = (1-\alpha)J_{n\times n}$, $C = A_\alpha(H) + \alpha\ (n+1)I_m$ and $D = (1-\alpha)I_m$. Here the matrix J means the matrix with all entries are 1.

The above matrix is equivalent to

$$A_\alpha(G\blacktriangledown H) = \begin{pmatrix} A_{n\times n} & B_{n\times m} & 0_{n\times n} & 0_{n\times m}\\ B^T_{m\times n} & C_{m\times m} & 0_{m\times n} & D_{m\times m}\\ 0_{n\times n} & 0_{n\times m} & A_{n\times n} & B_{n\times m}\\ 0_{m\times n} & D_{m\times m} & B^T_{m\times n} & C_{m\times m}\end{pmatrix} = \begin{pmatrix} M_1 & M_2\\ M_2 & M_1\end{pmatrix}$$

where $M_1 = \begin{pmatrix} A_{n\times n} & B_{n\times m}\\ B^T_{m\times n} & C_{m\times m}\end{pmatrix}$ and $M_2 = \begin{pmatrix} 0_{n\times n} & 0_{n\times m}\\ 0_{m\times n} & D_{m\times m}\end{pmatrix}$ By Lemma 4.2 in order to find the eigenvalues of $M_1 + M_2$ and $M_1 - M_2$. Now,

$$P_{M_1+M_2}(x) = Det\begin{vmatrix}(x-\alpha m)I_n - A_\alpha(G) & -(1-\alpha)J_{n\times m}\\ -(1-\alpha)J_{m\times n} & (x-\alpha(n+1)I_m - A_\alpha(H)\end{vmatrix}$$

Using Schur complement, we get

$$\begin{aligned}P_{M_1+M_2}(x) &= Det((x-\alpha(n+1)I_m - A_\alpha(H))\\ &\quad Det((x-\alpha m)I_n - A_\alpha(G) - (1-\alpha)^2 J_{m\times n}(x-\alpha(n+1)I_m - A_\alpha(H))^{-1}J_{n\times m}\\ &= P_{A_\alpha(H)}(x-\alpha(n+1))P_{A_\alpha(G)}(x-\alpha m)\\ &\quad \big((1-(1-\alpha)^2\Gamma_{A_\alpha(G)}(x-\alpha(n+1))I_m^T((x-\alpha n)I_m - A_\alpha(H))^{-1}I_m)\big)\\ &= P_{A_\alpha(H)}(x-\alpha(n+1))P_{A_\alpha(G)}(x-\alpha m)\\ &\quad \big((1-(1-\alpha)^2\Gamma_{A_\alpha(G)}(x-\alpha(n+1))\Gamma_{A_\alpha(H)}(x-\alpha m)\big)\end{aligned}$$

Using the above computation procedure, we can similarly find the following expression as

$$\begin{aligned}P_{M_1-M_2}(x) = {} & P_{A_\alpha(H)}(x+\alpha(n+1))P_{A_\alpha(G)}(x-\alpha m)\\ & \big((1-(1-\alpha)^2\Gamma_{A_\alpha(G)}(x+\alpha(n+1))\Gamma_{A_\alpha(H)}(x-\alpha m)\big)\end{aligned}$$

Therefore,

$$P_{A_\alpha(G\blacktriangledown H)}(x) = (P_{M_1+M_2}(x))(P_{M_1-M_2}(x))$$

Therefore,

$$
\begin{aligned}
P_{A_\alpha(G\blacktriangledown H)}(x) = {} & P_{A_\alpha(H)}(x-\alpha(n+1))P_{A_\alpha(G)}(x-\alpha m) \\
& \left((1-(1-\alpha)^2\Gamma_{A_\alpha(G)}(x-\alpha(n+1))\Gamma_{A_\alpha(H)}(x-\alpha m)\right) P_{A_\alpha(H)}(x+\alpha(n+1)) \\
& P_{A_\alpha(G)}(x-\alpha m)\left((1-(1-\alpha)^2\Gamma_{A_\alpha(G)}(x+\alpha(n+1))\Gamma_{A_\alpha(H)}(x-\alpha m)\right)
\end{aligned}
$$

□

As an application, using Theorem 4.1 we can construct infinite family of A_α-cospectral graph using Indu-Bala product

Corollary 4.1.1. *Let G_1 and G_2 are A_α-cospectral regular graph and H is any graph. Then $G_1\blacktriangledown H$ and $G_1\blacktriangledown H$ are A_α-cospectral.*

Corollary 4.1.2. *Let G_1 be a regular graph H_1 and H_2 are A_α-cospectral graphs with $\Gamma_{A_\alpha H_1}(x) = \Gamma_{A_\alpha H_1}(x)$. Then $G_1\blacktriangledown H_1$ and $G_1\blacktriangledown H_2$ are A_α-cospectral.*

5. Conclusion

The Indu-Bala product of graphs holds potential for modeling complex interconnected systems where relationships between two distinct groups of entities need to be analyzed. This framework can represent scenarios such as collaborations between companies, where different departments within each organization maintain robust internal connections while simultaneously engaging with specific counterparts in the partner company. For instance, a company's internal network could interact with that of its partner through designated links between corresponding departments, creating a structure with strong intra-group connections and additional inter-group relationships.

Such modeling captures the dual nature of internal cohesion and external collaboration, making the Indu-Bala product particularly suitable for studying such interactions. While the analysis of this graph product is still in its nascent stages, ongoing research and increasing interest in its properties suggest that it may find broader real-world applications in the near future.

In this study, we have explored different geodetic and domination variants in the Indu-Bala product of graphs. A similar kind of study can be done with respect to edge geodetic and edge domination variants. Also, we initiated the study on A_α spectrum of Indu-Bala product of graphs.

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