Journal of Finsler Geometry and its Applications
Vol. 6, No. 1 (2025), pp 76-82
https://doi.org/10.22098/jfga.2025.16485.1145

Closed graph property and Khalimsky spaces

Mehrnaz Pourattar[a], Fatemah Ayatollah Zadeh Shirazi[b*], and Mohammad Reza Mardanbeigi[c]

[a]Department of Mathematics, Science and Research Branch, Islamic Azad University, Tehran, Iran
[b]Faculty of Mathematics, Statistics and Computer Science, College of Science, University of Tehran, Tehran, Iran
[c]Department of Mathematics, Science and Research Branch, Islamic Azad University, Tehran, Iran

E-mail: mpourattar@yahoo.com
E-mail: f.a.z.shirazi@ut.ac.ir
E-mail: mrmardanbeigi@srbiau.ac.ir

Abstract. In the following text for Khalimsky $n-$dimensional space $\mathcal{K}^n$ we show self–map $f:\mathcal{K}^n\to\mathcal{K}^n$ has closed graph if and only if there exist integers $\lambda_1,\dots,\lambda_n$ such that f is a constant map with value $(2\lambda_1,\cdots,2\lambda_n)$. We also show each self–map on Khalimsky circle and Khalimsky sphere which has closed graph is a constant map.

Keywords: Alexandroff space, Closed graph, Digital topology, Khalimsky spaces.

1. Introduction

Alexandroff spaces introduced as “Diskrete Räume” by P. Alexandroff in [1]. One may consider different sub–classes of Alexandroff spaces like finite topological spaces [12], functional Alexandroff topological spaces [2], locally finite

*Corresponding Author
AMS 2020 Mathematics Subject Classification: 54C35, 54D05

topological spaces [7, 13], $n-$dimensional Khalimsky spaces [6, 7], etc. Interaction between Alexandroff topological spaces and other mathematical structures has been studied in different texts (see e.g. [5]).

Closed graph property has been studied in different categories of topological spaces: Banach spaces, complete metric spaces, and general topological spaces, etc. (see e.g. [9, 10, 14, 15, 17]).

Let's recall that in topological spaces X, Y and map $f : X \to Y$ we call $G_f := \{(x, f(x)) : x \in X\}$ the graph of f and we say $f : X \to Y$ has closed graph if G_f is a closed subset of $X \times Y$ (with product topology) [4]. The main aim of this text is to characterize all self–maps on $n-$dimensional Khalimsky spaces with closed graph.

Background on Alexandroff spaces. A topological space (X, τ) or simply X is an Alexandroff space if for each nonempty collection of open subsets of X like Γ, $\bigcap \Gamma$ is open too. In particular in Alexandroff space X for each $x \in X$ the intersection of all open neighbourhoods of X is the smallest open neighbourhood of x, we denote the smallest open neighbourhood of x by $V_X(x)$ or simply by $V(x)$.

Moreover, product of two topological Alexandroff spaces is an Alexandroff space too. Note that if X, Y are Alexandroff spaces and $(x, y) \in X \times Y$, then $V_{X\times Y}(x, y) = V_X(x)\times V_Y(y)$ (where $X\times Y$ is equipped with product topology). Using induction product of finitely of finitely many Alexandroff spaces is an Alexandroff space.

Background on Khalimsky spaces. Let's equip $\mathbb{Z} = \{0, \pm 1, \pm 2, \cdots\}$ with topology κ generated by basis $\{\{2n-1, 2n, 2n+1\} : n \in \mathbb{Z}\}\cup\{\{2n+1\} : n \in \mathbb{Z}\}$. We call the Alexandroff topological space $(\mathbb{Z}, \kappa)$ Khalimsky line and denote it by $\mathcal{K}$. For integers a, b with $a \leq b$ let

$\mathcal{A}$) $[a, b]_{\mathcal{K}} := \{x \in \mathbb{Z} : a \leq x \leq b\}$,
$\mathcal{B}$) $[a, +\infty)_{\mathcal{K}} := \{x \in \mathbb{Z} : x \leq a\}$,
$\mathcal{C}$) $(-\infty, a]_{\mathcal{K}} := \{x \in \mathbb{Z} : x \geq a\}$.

All subsets of $\mathcal{K}$ introduced in $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ and $\varnothing, \mathcal{K}$ are intervals of Khalimsky line $\mathcal{K}$.

For natural number n, we call $\mathcal{K}^n$ an $n-$dimensional Khalimsky space. $\mathcal{K}^2$ is called Khalimsky plane and one may consider it in digital topology too [11]. In topological space X suppose $\infty \notin X$ and let $A(X) := X\cup\{\infty\}$. Equip $A(X)$ with topology $\{U \subseteq X : U$ is an open subset of $X\} \cup \{U \subseteq A(X) : X \setminus U$ is a compact closed subset of $X\}$ then we call $A(X)$ one point compactification or Alexandroff compactification of X [16].

We call one point compactification of Khalimsy line as Khalimsky circle and also one point compactification of Khalimsky plane as Khalimsky sphere (see e.g., [6, 7]).

2. When does self–map $f : \mathcal{K}^n \to \mathcal{K}^n$ have closed graph?

In this section we investigate all cases which a self–map on $\mathcal{K}^n$ (resp. $A(\mathcal{K}^n)$) has closed graph.

Lemma 2.1. *For each $x, y \in \mathcal{K}^n$ there exists finite sequence $x = x_1, x_2, \ldots, x_p = y \in \mathcal{K}^n$ such that for each $i \in \{1, \ldots, p-1\}$ we have $V_{\mathcal{K}^n}(x_i) \cap V_{\mathcal{K}^n}(x_{i+1}) \neq \varnothing$.*

Proof. Let's use induction on n.

If $n = 1$ and $x, y \in \mathcal{K}$ choose $z, w \in \mathbb{Z}$ such that $|2z-x| \leq 1$ and $|2w-y| \leq 1$, hence $V_{\mathcal{K}}(x) \cap V_{\mathcal{K}}(2z) \neq \varnothing$ and $V_{\mathcal{K}}(2w) \cap V_{\mathcal{K}}(y) \neq \varnothing$. We may suppose $z \leq w$, then for $x_1 = x, x_2 = 2z, z_3 = 2z+2, \ldots, x_{w-z+2} = 2w, x_{w-z+3} = y$ and each $i \in \{1, \ldots, x-z+2\}$ we have

$$V_{\mathcal{K}}(x_i) \cap V_{\mathcal{K}}(x_{i+1}) \neq \varnothing.$$

Suppose the statement is true for $n = q \geq 1$, i.e. for each $a, b \in \mathcal{K}^q$ there exists finite sequence $a = x_1, x_2, \ldots, x_p = b \in \mathcal{K}^q$ such that for each $i \in \{1, \ldots, p-1\}$ we have

$$V_{\mathcal{K}^q}(x_i) \cap V_{\mathcal{K}^q}(x_{i+1}) \neq \varnothing.$$

Choose

$$x = (c, d), y = (u, v) \in \mathcal{K} \times \mathcal{K}^q (= \mathcal{K}^{q+1}),$$

since $c, u \in \mathcal{K}$ and $d, v \in \mathcal{K}^q$ using induction's hypothesis there exist $c_1 = c, \ldots, c_s = u \in \mathcal{K}$ and $d_1 = d, \ldots, d_t = v \in \mathcal{K}^q$ such that for each $i \in \{1, \ldots, s-1\}$ and $j \in \{1, \ldots, t-1\}$ we have

$$V_{\mathcal{K}}(c_i) \cap V_{\mathcal{K}}(c_{i+1}) \neq \varnothing, \quad V_{\mathcal{K}^q}(d_j) \cap V_{\mathcal{K}^q}(d_{j+1}) \neq \varnothing.$$

Let

$$x_1 = x = (c, d) = (c_1, d), x_2 = (c_2, d), \ldots, x_s = (c_s, d) = (u, d) = (u, d_1),$$
$$x_{s+1} = (u, d_2), \ldots, x_{s+t-1} = (u, d_t) = (u, v).$$

Then, for each $i \in \{1, \ldots, s+t-2\}$ we have $V_{\mathcal{K}^{q+1}}(x_i) \cap V_{\mathcal{K}^{q+1}}(x_{i+1}) \neq \varnothing$ (use the fact that $V_{\mathcal{K}^{q+1}}(c_k, d_l) = V_{\mathcal{K}}(c_k) \times V_{\mathcal{K}^q}(d_l)$. □

Lemma 2.2. *In topological space X if $f : X \to X$ has closed graph, then for each $x \in X$, and $y \in \overline{\{x\}}$ we have $f(x) = f(y)$, in particular if x has the smallest open neighbourhood like V, then for each $z \in V$, $f(x) = f(z)$.*

Proof. Suppose $f : X \to X$ has closed graph, $x \in X$ and $y \in \overline{\{x\}}$. Thus

$$(y, f(x)) \in \overline{\{x\}} \times \overline{\{f(x)\}} = \overline{\{(x, f(x))\}} \subseteq \overline{G_f} = G_f = \{w, f(w)) : w \in X\},$$

which leads to $f(y) = f(x)$. In order to complete the proof note that if x has the smallest open neighbourhood like V, then for each $z \in V$ we have $x \in \overline{\{z\}}$, therefore $f(z) = f(x)$. □

Theorem 2.3. *If $n \geq 1$ and X is one of the following spaces:*

- *an interval in $\mathcal{K}$,*
- *$n-$dimensional Khalimsky space $\mathcal{K}^n$,*
- *$A(\mathcal{K}^n)$,*

then each $f : X \to X$ with closed graph is a constant map.

Proof. Consider $f : X \to X$ has closed graph.
First suppose X is an interval in $\mathcal{K}$ with at least two elements or X is $\mathcal{K}^n$, then by Lemma 2.1 for each $x, y \in X$ there exist $x = x_1, \ldots, x_p = y \in X$ such that for each $i \in \{1, \ldots, p-1\}$ we have $V_X(x_i) \cap V_X(x_{i+1}) \neq \varnothing$. By Lemma 2.2, $f(x) = f(y)$, which shows that f is a constant map.

Now Suppose $X = A(\mathcal{K}^n)$, using a similar method described in the previous paragraph the restriction of f to $\mathcal{K}^n$ is constant. Suppose $f(x) = c$ for all $x \in \mathcal{K}^n$, then

$$(\infty, c) \in A(\mathcal{K}^n) \times \{c\} = \overline{\mathcal{K}^n \times \{c\}} \subseteq \overline{G_f} = G_f .$$

Hence $f(\infty) = c$ and f is a constant map in this case too. □

Note 2.4. *In nonempty topological spaces X, Y if constant map $f : \underset{x \mapsto c}{X \to Y}$ has closed graph, then $X \times \{c\} = G_f = \overline{G_f} = \overline{X \times \{c\}} = X \times \overline{\{c\}}$, hence $\overline{\{c\}} = \{c\}$ and c is a closed point of X. Thus constant map $X \underset{x \mapsto b}{\to} Y$ has closed graph if and only if b is a closed point of X.*

Now we are ready to establish our main theorem.

Theorem 2.5 (main)**.** *Consider $n \geq 1$.*

- *Suppose X is an interval in $\mathcal{K}$ with at least two elements. $f : X \to X$ has closed graph if and only if there exists even integer $2\lambda \in X$ such that f is the constant map with value 2λ.*

- *$f : \mathcal{K}^n \to \mathcal{K}^n$ has closed graph if and only if there exist even integers $2\lambda_1, \ldots, 2\lambda_n$ such that f is the constant map with value $(2\lambda_1, \ldots, 2\lambda_n)$.*

- *$f : A(\mathcal{K}^n) \to A(\mathcal{K}^n)$ has closed graph if and only if one of the following conditions occurs:*
  - *there exist even integers $2\lambda_1, \ldots, 2\lambda_n$ such that f is the constant map with value $(2\lambda_1, \ldots, 2\lambda_n)$,*
  - *f is the constant map with value ∞.*

Proof. If X is an interval in $\mathcal{K}$ with at least two elements, then $X \cap 2\mathbb{Z}$ is the collection of all closed points of X. Also $(2\mathbb{Z})^n$ is the collection of all closed points of $\mathcal{K}^n$. Moreover $(2\mathbb{Z})^n \cup \{\infty\}$ is the collection of all closed points of $A(\mathcal{K}^n)$. Now use Theorem 2.3 and Note 2.5 to complete the proof. $\square$

3. A Diagram

In this section via a diagram we compare the collections of self–maps on $\mathcal{K}^n$ satisfying one of the following conditions:

- has closed graph,
- is constant,
- is continuous,
- is quasi–continuous,
- is just a self–map on $\mathcal{K}^n$.

Let's recall that in topological spaces X, Y, $f : X \to Y$ is quasi–continuous at $a \in X$ if for each open neighbourhoods U of a and V of $f(a)$ there exists nonempty open subset W of U such that $f(W) \subseteq V$. Also we say $f : X \to Y$ is quasi–continuous if it is quasi–continuous at all pints of X [3, 8].
Now we have the following diagram:

The collection of all self–maps $f : \mathcal{K}^n \to \mathcal{K}^n$

The collection of all quasi–continuous maps $f : \mathcal{K}^n \to \mathcal{K}^n$

The collection of all continuous maps $f : \mathcal{K}^n \to \mathcal{K}^n$

The collection of all constant maps $f : \mathcal{K}^n \to \mathcal{K}^n$

The collection of all maps $f : \mathcal{K}^n \to \mathcal{K}^n$ with closed graph

Example 3.1

Example 3.2

Example 3.3

Example 3.4

Example 3.5

In order to complete the above diagram consider the following examples which have been referred to the diagram.

Example 3.1. *Consider* $f : \mathcal{K}^n \to \mathcal{K}^n$ *with* $f(x) = (0, \cdots, 0)$ *for all* $x \in \mathcal{K}^n$*, then by Theorem 2.5,* $f : \mathcal{K}^n \to \mathcal{K}^n$ *has closed graph. Note that by Theorem 2.5 all self–maps on* $\mathcal{K}^n$ *with closed graph are constant maps.*

Example 3.2. *Consider constant map* $f : \mathcal{K}^n \to \mathcal{K}^n$ *with* $f(x) = (1, \cdots, 1)$ *for all* $x \in \mathcal{K}^n$ *then by Theorem 2.5,* $f : \mathcal{K}^n \to \mathcal{K}^n$ *does not have closed graph. Note that all constant maps are continuous.*

Example 3.3. $f : \mathcal{K}^n \to \mathcal{K}^n$ *with* $f(x) = x$ *for all* $x \in \mathcal{K}^n$ *is a continuous non–constant map. Note that all continuous maps are quasi–continuous.*

Example 3.4. *Consider* $h : \mathcal{K} \to \mathcal{K}$ *with* $h(2m) = h(2m + 1) = 2m + 1$ *for each* $m \in$ *, then* $f : \mathcal{K}^n \to \mathcal{K}^n$ *with* $f(x_1, \cdots, x_n) = (h(x_1), \cdots, h(x_n))$ *for all* $(x_1, \cdots, x_n) \in \mathcal{K}^n$ *is a quasi–continuous map which is not continuous.*

Example 3.5. *Consider* $h : \mathcal{K} \to \mathcal{K}$ *with* $h(2m) = -h(2m + 1) = 1$ *for each* $m \in$ *, then* $f : \mathcal{K}^n \to \mathcal{K}^n$ *with* $f(x_1, \cdots, x_n) = (h(x_1), \cdots, h(x_n))$ *for all* $(x_1, \cdots, x_n) \in \mathcal{K}^n$ *is a is not quasi–continuous.*

Acknowledgment: The authors wish to dedicate this paper to our lady, Fatimah-Zahra (as).

References

1. P. Alexandroff, *Diskrete Räume*, Mat. Sb. **2** (1937), 501–518.
2. F. Ayatollah Zadeh Shirazi and N. Golestani, *Functional Alexandroff spaces*, Hacettepe Journal of Mathematics and Statistics **40/4** (2011), 515–522.
3. F. Ayatollah Zadeh Shirazi and N. Shirinbayan, *The size of quasicontinuous maps on Khalimsky line*, J. Finsler Geom. Appl. **4 no. 2** (2023), 38–42.
4. T. Banakh, M. Filipczak, and J. Wódka, *Returning functions with closed graph are continuous*, Mathematica Slovaca, **70 no. 2** (2020), 297–304.
5. R. Berghammer and M. Winter, *Order- and graph-theoretic investigation of dimensions of finite topological spaces and Alexandroff spaces*, Monatsh. Math. **190 no. 1** (2019), 33–78.
6. S.-E. Han, *Semi–separation axioms of the infinite Khalimsky topological sphere*, Topology Appl. **275** (2020), 107006.
7. S.-E. Han, Sang-Eon, and I.-K. Na, *Topologies associated with the one point compactifications of Khalimsky topological spaces*, Topology Appl. **241** (2018), 333–344.
8. L. Holá and D. Holý, *Metrizability of the space of quasicontinuous functions*, Topology and its Applications **246** (2018), 137–143.
9. G. B. Folland, *Real analysis. Modern techniques and their applications*, Second edition. Pure and Applied Mathematics, A Wiley-Interscience Publication, John Wiley & Sons, Inc., 1999.
10. T. Husain, *The open mapping and closed graph theorems in topological vector spaces*, Clarendon Press, Oxford, 1965.
11. E. Khalimsky, R. Kopperman, and P. R. Meyer, *Computer graphics and connected topologies on finite ordered sets*, Topology Appl. **36 no. 1** (1990), 1–17.
12. V. Krishnamurthy, *On the number of topologies on a finite set*, American Mathematical Monthly **73/2** (1966), 154–57.

13. J. R. Munkres, *Topology A first course*, Prentice–Hall Inc., 1975.
14. D. Noll, *Topological spaces satisfying a closed graph theorem*, Topology Appl. **349** (2024), 108903.
15. B. J. Pettis, *Closed graph and open mapping theorems in certain topologically complete spaces*, Bull. London Math. Soc. **6** (1974), 37–41.
16. L. A. Steen and J. A. Seebach, *Counterexamples in topology*, Holt, Rinehart and Winston, Inc., 1970.
17. J. H. van der Walt, *A closed graph theorem for order bounded operators*, Quaest. Math. **39 no. 2** (2016), 167–178.

Received: 04.01.2025
Accepted: 17.03.2025