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Eigenvalue estimate for the Laplace operator on Finsler manifold with weighted Ricci curvature

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Abstract. In this paper, we study about the first eigenvalue of the bi-Laplace operator on Finsler manifolds. Considering a bounded weighted Ricci curvature on a complete Finsler manifold, we obtain an upper bound for the first eigenvalues of Buckling and Clamped plate problems related with the first eigenvalue of the Laplace operator.

Keywords: Eigenvalue problem, Finsler manifold, Weighted Ricci curvature.

1. Introduction

Studying eigenvalues and eigenfunctions of geometric operators play an important role in global differential geometry. These studies make connection between geometry and analysis of the manifold. So far, eigenvalue estimate has been extensively studied on Riemannian manifolds for different geometric operators, such as, Laplace, p-Laplace and bi-Laplace. According to the importance of eigenvalues, authors have been tried to find some relations for different eigenvalues(see [5, 7, 8, 9]). Due to the fact that Finsler geometry is

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a natural generalization of Riemannian geometry, in recent decades eigenvalue estimate have been widely investigated for Finsler operators. For example see [3, 12, 13, 14] for some interesting results on Finsler p-Laplacian. Following some methods of recent researches on Riemannian manifolds, we are going to answer below questions:

1- Is there any relations between the first eigenvalue of Laplace operator and the first eigenvalue of Clamped plate problem?

2- Is there any relations between the first eigenvalue of Laplace operator and the first eigenvalue of Buckling problem?

There are two well-known bi-Laplace eigenvalue problems define on a Riemannian manifold as below:

$$\begin{cases} \Delta^2 u = \Gamma u \qquad in \qquad N, \\ u|_{\partial B} = \dfrac{\partial u}{\partial \overrightarrow{n}}|_{\partial N} = 0 \quad \text{(Clamped plate)} \end{cases}$$

and

$$\begin{cases} \Delta^2 u = -\Lambda \Delta u \qquad in \qquad N, \\ u|_{\partial B} = \dfrac{\partial u}{\partial \overrightarrow{n}}|_{\partial N} = 0 \quad \text{(Buckling)} \end{cases}$$

where Δ^2 is the bi-harmonic operator, B is a bounded domain of N, and $\overrightarrow{n}$ denotes the outer unit normal vector field of ∂N. We refer the readers to [1, 4, 6] for studying eigenvalue estimate of these two problems.

Similarly, for a compact connected Finsler manifold $(M, F, d\mu)$ with smooth boundary ∂M, consider $M_u = \{x \in M | du(x) \neq 0\}$, then Clamped plate and Buckling problems define as follows:

$$\begin{cases} \Delta^{\nabla u} \Delta u = \Gamma u \qquad in \qquad M_u, \\ u|_{\partial B} = g_{\overrightarrow{n}}(\overrightarrow{n}, \nabla u)|_{\partial M} = 0 \end{cases}$$

and

$$\begin{cases} \Delta^{\nabla u} \Delta u = -\Lambda \Delta u \qquad in \qquad M_u, \\ u|_{\partial B} = g_{\overrightarrow{n}}(\overrightarrow{n}, \nabla u)|_{\partial M} = 0 \end{cases}$$

here Δ and $\Delta^{\nabla u}$ are Laplacian and weighted Laplacian, the outer unit normal vector field on ∂M denotes by $\overrightarrow{n}$, which induces the Riemannian structure $g_{\overrightarrow{n}}$ on ∂M.

Consider $H^2(M) = \{u : u, F(\nabla u), |\nabla^2 u|^2_{HS(\nabla u)} \in L^2(M)\}$, here

$$\int_M |\nabla^2 u|^2_{HS(\nabla u)} d\mu := \int_{M_u} |\nabla^2 u|^2_{HS(\nabla u)} d\mu.$$

Let $H_0^2(M)$ be the subset of $H^2(M)$ defines

$$H_0^2(M) = \left\{ u \in H^2(M) : u|_{\partial M} = \frac{\partial u}{\partial \overrightarrow{n}}|_{\partial M} = 0 \right\}.$$

The first eigenvalue of Clamped plate and Buckling define as follows (see [11]):

$$\Gamma_1 = \min_{u\in H_0^2(M),u\neq 0} \frac{\int_M (\Delta u)^2 d\mu}{\int_M u^2 d\mu},$$

and

$$\Lambda_1 = \min_{u\in H_0^2(M),u\neq 0} \frac{\int_M (\Delta u)^2 d\mu}{\int_M (F(\nabla u))^2 d\mu}$$

2. **Preliminaries**

In this section, we will review some necessary formula in finsler geometry. (For further reading we refer to [2])

Definition 2.1. *Let M be an n-dimensional smooth manifold. We say a function $F : TM \to [0,\infty)$ is a Finsler structure if the following three conditions hold:*
(i) F is C^∞ on $TM \setminus \{0\}$;
(ii) $F(\lambda V) = \lambda F(V)$, $\forall V \in TM \setminus \{0\}$, and $\lambda > 0$;
(iii) For any $V \in T_xM \setminus \{0\}$, the $n \times n$ matrix

$$(g_{ij}(V))_{ij=1}^n = \big(\frac{1}{2}[F^2(V)]_{V^iV^j}\big)_{ij=1}^n$$

is positive definite. The pair (M,F) is called a Finsler manifold.

A Finsler manifold is said to be reversible if $F(-V) = F(V)$ for all $V \in TM\setminus\{0\}$. A triple $(M,F,d\mu)$ constituted with a smooth, connected n-dimensional manifold M, a Finsler structure F on M, and a measure μ on M is called a Finsler measure space. For every non-vanishing vector field V, the positive definite matrix $(g_{ij}(V))_{ij=1}^n$ induces a Riemannian structure g_V of T_xM with

$$g_V\left(\Sigma_{i=1}^n X^i \frac{\partial}{\partial x^i}|_x, \Sigma_{j=1}^n Y^j \frac{\partial}{\partial x^j}|_x\right) = \Sigma_{ij=1}^n g_{ij}(V)X^iY^j, for \quad X,Y \in T_xM.$$

In particular, $g_V(V,V) = F(V)^2$.

Consider the natural projection map $\pi : TM \to M$, the pull-back bundle π^*TM admit a unique linear connection as Chern connection. The Chern connection is determined by the following structure equations:
1) Torsion freeness:

$$D_X^V Y - D_Y^V X = [X,Y],$$

2) Almost g-compatibility:

$$Xg_V(Y,Z) = g_V(D_X^V Y, Z) + g_V(Y, D_X^V z) + 2C_V(D_X^V V, Y, Z),$$

here $V \in T_xM \setminus \{0\}$, $X,Y,Z \in TM$, and

$$C_V(X,Y,Z) = C_{ijk}(V)X^iY^jZ^k = \frac{1}{4}\frac{\partial^3 F^2}{\partial V^i \partial V^j \partial V^k}(V)X^iY^jZ^k,$$

denotes the Cartan tensor and $D^V_X Y$ the covariant derivative with respect to reference vector $V \in T_xM \setminus \{0\}$.

The coefficient of the Chern connection are

$$\Gamma^i_{jk} = \frac{1}{2} g^{il}\left(\frac{\delta g_{kl}}{\delta x^j} + \frac{\delta g_{jl}}{\delta x^k} - \frac{\delta g_{jk}}{\delta x^l}\right),$$

that compute as follows

$$D^V_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \Gamma^i_{jk}(x,V)\frac{\partial}{\partial x^k},$$

where

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^j_i \frac{\partial}{\partial y^j},\ N^j_i = \frac{\partial G^j}{\partial y^i},\ G^j = \frac{1}{4} g^{jl}\{[F^2]_{x^k y^l} y^k - [F^2]_{x^l}\}.$$

Now, we recall the notion of the Legendre transform defined as follows:

$$\mathcal{L}(V) = \begin{cases} g_V(V,.) \in T^*_xM,\ V \in T_xM \setminus \{0\}, \\ 0, \qquad\qquad V = 0. \end{cases}$$

It can be verified that $F(V) = F^*(\mathcal{L}(V))$ for any $V \in TM$, where F^* is the dual norm of Finsler structure F on the cotangent space T^*M, defined

$$F^*(x,\xi) = \sup_{F(x,V) \leq 1} \xi(V) \quad for\, any\ \ \xi \in T^*M.$$

Let $u : M \to \mathbb{R}$ be a smooth function on M, $M_u := \{x \in M | du(x) \neq 0\}$. The gradient of u is

$$\nabla u(x) = \mathcal{L}^{-1}(du(x)) \in T_xM,$$

and the Hessian of u using Chern connection can be written as

$$\nabla^2 u(X,Y) = g_{\nabla u}(D^{\nabla u}_X \nabla u, Y).$$

Let $(M,F,d\mu)$ be a Finsler measure space, and $d\mu = \sigma(x)dx$ the volume form on M. For any smooth vector field $V \in TM$, the divergence of V is defined by

$$\operatorname{div}(V) = \Sigma^n_{i=1}\left(\frac{\partial V^i}{\partial x^i} + V^i \frac{\partial \log \sigma}{\partial x^i}\right).$$

For any smooth function u on M, the Finsler Laplacian is as follows

$$\Delta u = \operatorname{div}(\nabla u) = g^{ij}(\nabla u)\left(\frac{\partial^2 u}{\partial x^i \partial x^j} - \Gamma^k_{ij}(\nabla u)\frac{\partial u}{\partial x^k} + \frac{\partial u}{\partial x^i}\frac{\partial \log \sigma}{\partial x^i}\right).$$

Consider $(M,F,d\mu)$ as a Finsler measure space with boundary ∂M, then we shall view ∂M as a hypersurface embedded in M. Also ∂M is a Finsler manifold with a Finsler structure $F_{\partial M}$ induced by F. For any $x \in \partial M$, there is exactly two unit normal vectors $\overrightarrow{n}$, which are characterized by

$$T_x(\partial M) = \Big\{V \in T_xM | g_{\overrightarrow{n}}(\overrightarrow{n}, V) = 0, g_{\overrightarrow{n}}(\overrightarrow{n}, \overrightarrow{n}) = 1\Big\}.$$

In this paper, we choose the normal vector that points outward M. Note that, if $\overrightarrow{n}$ be a normal vector, then $-\overrightarrow{n}$ may not be a normal vector unless F be reversible. The normal vector $\overrightarrow{n}$ induces a volume form $d\mu_{\overrightarrow{n}}$ on ∂M from $d\mu$ by

$$V \rfloor d\mu = g_{\overrightarrow{n}}(\overrightarrow{n}, V) d\mu_{\overrightarrow{n}}, \ \ \forall V \in TM. \tag{2.1}$$

In [8], it is shown that the Stokes theorem holds as follows

$$\int_M \operatorname{div}(V) d\mu = \int_{\partial M} g_{\overrightarrow{n}}(\overrightarrow{n}, V) d\mu_{\overrightarrow{n}}.$$

Given two linearly independent vectors $V, W \in T_xM \setminus \{0\}$, the flag curvature is defined by

$$K(V, W) := \frac{g_V(R^V(V, W)W, V)}{g_V(V, V)g_V(W, W) - g_V(V, W)^2},$$

where R^V is the Chern curvature

$$\mathbf{R}^V(X, Y)Z = \nabla_X^V \nabla_Y^V Z - \nabla_Y^V \nabla_X^V Z - \nabla_{[X,Y]}^V Z.$$

Then the Ricci curvature of V for (M, F) is

$$\mathbf{Ric}(V) = \sum_{i=1}^{n-1} K(V, e_i),$$

here $e_1, \cdots, e_{n-1}, \dfrac{V}{F(V)}$ form an orthonormal basis of T_xM with respect to g_V, namely, one has $\operatorname{Ric}(\lambda V) = \operatorname{Ric}(V)$ for any $\lambda > 0$.

For a given volume form $d\mu = \sigma(x)dx$ and a vector $y \in T_xM \setminus \{0\}$, the distortion of $(M, F, d\mu)$ is defined by

$$\tau(V) := \ln \frac{\sqrt{\det(g_{ij}(y))}}{\sigma}.$$

Considering the rate of changes of the distortion along geodesics, leads to the so-called S-curvature as follows

$$\mathbf{S}(V) := \frac{d}{dt}[\tau(\gamma(t), \dot{\gamma}(t)]_{t=0},$$

where $\gamma(t)$ is the geodesic with $\gamma(0) = x$ and $\dot{\gamma}(0) = V$.

The weighted Ricci curvature on Finsler manifolds, which was introduced by Ohta [9], motivated by the work of Lott-Villani in [3] and Sturm [12]. It is defined as follows:

Definition 2.2. [9] *Let $(M, F, d\mu)$ be a Finsler n-manifold with volume form $d\mu$. Given a vector $V \in T_xM$, let $\gamma : (-\varepsilon, \varepsilon) \to M$ be a geodesic with $\gamma(0) = x$, $\dot{\gamma}(0) = V$. Define*

$$\dot{\boldsymbol{S}}(V) := F^{-2}(V) \frac{d}{dt}\Big[\boldsymbol{S}(\gamma(t), \dot{\gamma}(t)\Big]_{t=0}.$$

Then the weighted Ricci curvatures of M defined as follows

$$\begin{aligned} Ric_n(V) &:= \begin{cases} Ric(V)+\dot{S}(V), & for\ S(V)=0,\\ -\infty, & otherwise, \end{cases}\\ Ric_N(V) &:= Ric(V)+\dot{S}(V)-\frac{S(V)^2}{(N-n)F(V)^2},\ \ \forall N\in(n,\infty),\\ Ric_\infty(V) &:= Ric(V)+\dot{S}(V). \end{aligned}$$

Lemma 2.3 ([13])**.** *Let $(M,F,d\mu)$ be a Finsler measure space and $u:M\to\mathbb{R}$ be a smooth function on M. Then on $M_u=\{x\in M:\nabla u|_x\neq 0\}$, we have*

$$\Delta u=\Sigma_{i=1}^n u_{ii}-S(\nabla u),$$

where $u_{ii}=g_{\nabla u}(\nabla^2 u(e_i,e_i))$, $\{e_i\}_{i=1}^n$ is the local orthonormal frame with respect to $g_{\nabla u}$ on M_u.

We state Bochner-Weitzenböck formula from [10]:

Lemma 2.4. *Let $(M,F,d\mu)$ be a Finsler measure space, and $u:M\to\mathbb{R}$ a smooth function on M. Then*

$$\Delta^{\nabla u}\left(\frac{F(\nabla u)^2}{2}\right)-D(\Delta u)(\nabla u)=\mathrm{Ric}_\infty(\nabla u)+|\nabla^2 u|^2_{HS(\nabla u)}, \tag{2.2}$$

as well as

$$\Delta^{\nabla u}\left(\frac{F(\nabla u)^2}{2}\right)-D(\Delta u)(\nabla u)\geq \mathrm{Ric}_N(\nabla u)+\frac{(\Delta u)^2}{N}, \tag{2.3}$$

for $N\in(n,\infty)$, point-wise on M_u.

3. **Main Results**

Theorem 3.1. *Let $(M^n,F,d\mu)$ be a complete Finsler manifold and* Ric_N *satisfy*

$$\mathrm{Ric}_N(\nabla u)+\frac{2\sqrt{3}}{3}\frac{\nabla^2 u(\nabla u,\nabla u)}{u}\geq 0, \tag{3.1}$$

where u is the first Dirichlet eigenfunction on a bounded domain $\Omega\subset M^n$ corresponding to λ_1. Then, we obtain

$$\Lambda_1\leq\frac{7+\sqrt{12}}{3}\lambda_1.$$

Theorem 3.2. *For a complete Finsler manifold $(M,F,d\mu)$ with*

$$\mathrm{Ric}_N(\nabla u)+2(c-2)\frac{\nabla^2 u(\nabla u,\nabla u)}{u}\geq 0, \tag{3.2}$$

where u is the first Dirichlet eigenfunction on $B\subset M$, corresponding to λ_1. Then

$$\Gamma_1\leq f(c)\lambda_1^2. \tag{3.3}$$

Here $c \approx 2.4$ such that $f(c)$ attains the minimum value

$$f(c) = \frac{c^2}{2c-1}\left[1 + \frac{1}{3}\left(\frac{(3c-4)+1}{\sqrt{2(3c-4)}-1}\right)^2\right].$$

Proving our main result, first we need to prove the following Lemma:

Lemma 3.3. *Let $(M, F, d\mu)$ be a complete Finsler manifold. For any constant $c > 2$, if*

$$\mathrm{Ric}_N(\nabla u) + 2(c-2)\frac{\nabla^2 u(\nabla u, \nabla u)}{u} \geq 0, \tag{3.4}$$

where u is a first Dirichlet eigenfunction on $B \subset M$ corresponding to λ_1, then

$$(2c - \frac{7}{3}) - \frac{2}{3}\sqrt{2(3c-4)} \leq \frac{\lambda_1}{I_c} \leq (2c - \frac{7}{3}) + \frac{2}{3}\sqrt{2(3c-4)}. \tag{3.5}$$

Proof. Due to the definition of the first Dirichlet eigenvalue, we know $\Delta u = -\lambda_1 u$ on B such that $u|_{\partial B} = 0$. Multiplying both sides to u^{2c-1}, we derive

$$\int_B u^{2c-2} F(\nabla u)^2 d\mu = \frac{1}{2c-1}\lambda_1 \int_B u^{2c} d\mu. \tag{3.6}$$

Let I_c be the following function

$$I_c = \frac{\int_B u^{2c-4} F(\nabla u)^4 d\mu}{\int_B u^{2c-2} F(\nabla u)^2 d\mu}. \tag{3.7}$$

Combining (3.5) with (3.6), we have

$$\frac{1}{2c-1}\lambda_1 I_c = \frac{\int_B u^{2c-4} F(\nabla u)^4 d\mu}{\int_B u^{2c} d\mu}. \tag{3.8}$$

On the other hand

$$\begin{aligned}
&\lambda_1 \int_B u^{2c-2} F(\nabla u)^2 d\mu \\
=\ & \int_B u^{2c-3} F(\nabla u)^2 (-\Delta u) d\mu \\
=\ & \int_B du(\nabla^{\nabla u}(u^{2c-3} F(\nabla u)^2)) d\mu \\
=\ & (2c-3)\int_B u^{2c-4} F(\nabla u)^4 d\mu + \int_B u^{2c-3} du(\nabla^{\nabla u}(F(\nabla u)^2)) d\mu. \qquad (3.9)
\end{aligned}$$

Now, by I_c, we can rewrite (3.9) as

$$\begin{aligned}
&\lambda_1 \int_B u^{2c-2} F(\nabla u)^2 d\mu \qquad (3.10)\\
&= (2c-3) I_c \int_B u^{2c-2} F(\nabla u)^2 d\mu + \int_B u^{2c-3} du(\nabla^{\nabla u}(F(\nabla u)^2)) d\mu,
\end{aligned}$$

So

$$(2c-3)I_c\int_B u^{2c-2}F(\nabla u)^2 d\mu \tag{3.11}$$
$$=\int_B(\lambda_1 u^{c-1}F(\nabla u)-2u^{c-2}du(\nabla^{\nabla u}F(\nabla u)^2)(u^{c-1}F(\nabla u))d\mu.$$

Using Hölder inequality, we obtain

$$\begin{aligned}
&(2c-3)^2 I_c^2\Big(\int_B u^{2c-2}F(\nabla u)^2 d\mu\Big)^2\\
=&\Big(\int_B(\lambda_1 u^{c-1}F(\nabla u)-2u^{c-2}du(\nabla^{\nabla u}F(\nabla u)))(u^{c-1}F(\nabla u))d\mu\Big)^2\\
\le&\int_B(\lambda_1 u^{c-1}F(\nabla u)-2u^{c-2}du(\nabla^{\nabla u}F(\nabla u))^2 d\mu\int_B u^{2c-2}F(\nabla u)^2 d\mu. \qquad (3.12)
\end{aligned}$$

Thus

$$\begin{aligned}
&(2c-3)^2 I_c^2\int_B u^{2c-2}F(\nabla u)^2 d\mu\\
\le&\int_B(\lambda_1 u^{c-1}F(\nabla u)-2u^{c-2}du(\nabla^{\nabla u}F(\nabla u)))^2 d\mu\\
=&\lambda_1^2\int_B u^{2c-2}F(\nabla u)^2 d\mu+4\int_B u^{2c-4}(du(\nabla^{\nabla u}F(\nabla u)))^2 d\mu\\
&-4\lambda_1\int_B u^{2c-3}F(\nabla u)du(\nabla^{\nabla u}F(\nabla u))d\mu. \qquad (3.13)
\end{aligned}$$

and

$$\begin{aligned}
&-4\lambda_1\int_B u^{2c-3}F(\nabla u)du(\nabla^{\nabla u}F(\nabla u))d\mu\\
=&-\frac{1}{c-1}\lambda_1\int_B\nabla u^{2c-2}\nabla^{\nabla u}(F(\nabla u)^2)d\mu\\
=&\frac{1}{c-1}\lambda_1\int_B F(\nabla u)^2\Delta(u^{2c-2})d\mu\\
=&2\lambda_1\int_B[(2c-3)u^{2c-4}F(\nabla u)^4-\lambda_1 u^{2c-2}F(\nabla u)^2]d\mu. \qquad (3.14)
\end{aligned}$$

Substituting (3.14) in (3.13), we get

$$\begin{aligned}
&(2c-3)^2 I_c^2\int_B u^{2c-2}F(\nabla u)^2 d\mu\\
\le&-\lambda_1^2\int_B u^{2c-2}F(\nabla u)^2 d\mu+2(2c-3)\lambda_1\int_B u^{2c-4}F(\nabla u)^4 d\mu\\
&+4\int_B u^{2c-4}(du(\nabla^{\nabla u}F(\nabla u)))^2 d\mu. \qquad (3.15)
\end{aligned}$$

Using Bochner-Weitzenböck formula (2.3), we have

$$\begin{aligned}
&\int_B u^{2c-4}F(\nabla u)^2|\nabla^2 u|^2 d\mu \\
\leq\ & \int_B u^{2c-4}F(\nabla u)^2\left[\Delta^{\nabla u}(\frac{F(\nabla u)^2}{2}) - \mathrm{Ric}_N(\nabla u) + \lambda F(\nabla u)^2\right]d\mu \\
=\ & \frac{1}{2}\int_B u^{2c-4}F(\nabla u)^2\Delta^{\nabla u}(F(\nabla u))^2 d\mu - \int_B u^{2c-4}F(\nabla u)^2\mathrm{Ric}_N(\nabla u)d\mu \\
& +\lambda_1\int_B u^{2c-4}F(\nabla u)^4 d\mu. \qquad (3.16)
\end{aligned}$$

Where

$$\begin{aligned}
&\frac{1}{2}\int_B u^{2c-4}F(\nabla u)^2\Delta^{\nabla u}(F(\nabla u))^2 d\mu \qquad (3.17)\\
&= -\frac{1}{2}\int_B u^{2c-4}|(\nabla^{\nabla u}F(\nabla u)^2|^2 d\mu \\
&-(c-2)\int_B u^{2c-5}F(\nabla u)^2(du(\nabla^{\nabla u}F(\nabla u)^2))d\mu
\end{aligned}$$

Putting (3.17) into (3.16), gives

$$\begin{aligned}
&\int_B u^{2c-4}F(\nabla u)^2|\nabla^2 u|^2 d\mu \\
\leq\ & -\frac{1}{2}\int_B u^{2c-4}|\nabla^{\nabla u}F(\nabla u)^2|^2 d\mu \\
& -(c-2)\int_B u^{2c-5}F(\nabla u)^2(du(\nabla^{\nabla u}F(\nabla u)^2))d\mu \\
& -\int_B u^{2c-4}F(\nabla u)^2\mathrm{Ric}_N(\nabla u)d\mu + \lambda_1\int_B u^{2c-4}F(\nabla u)^4 d\mu \\
=\ & -2\int_B u^{2c-4}(du(\nabla^{\nabla u}F(\nabla u)))^2 d\mu \\
& -(c-2)\int_B u^{2c-5}F(\nabla u)^2(du(\nabla^{\nabla u}F(\nabla u)^2))d\mu \\
& -\int_B u^{2c-4}F(\nabla u)^2\mathrm{Ric}_N(\nabla u)d\mu + \lambda_1\int_B u^{2c-4}F(\nabla u)^4 d\mu. \qquad (3.18)
\end{aligned}$$

Using (3.4) in (3.18), we obtain

$$\begin{aligned}
\int_B u^{2c-4}F(\nabla u)^2|\nabla^2 u|^2 d\mu \leq\ & -2\int_B u^{2c-4}(du(\nabla^{\nabla u}F(\nabla u)))^2 d\mu \\
& +\lambda_1\int_B u^{2c-4}F(\nabla u)^4 d\mu, \qquad (3.19)
\end{aligned}$$

By Kato's inequality it follows

$$F(\nabla u)^2|\nabla^2 u|^2 \geq F(\nabla u)^2|\nabla^{\nabla u}F(\nabla u)|^2 \geq (du(\nabla^{\nabla u}F(\nabla u)))^2,$$

which gives

$$\int_B u^{2c-4}(du(\nabla^{\nabla u}F(\nabla u)))^2 d\mu \leq \frac{1}{3}\lambda_1 \int_B u^{2c-4}F(\nabla u)^4 d\mu. \tag{3.20}$$

From (3.20), (3.15) changes as

$$\begin{aligned} & (2c-3)^2 I_c^2 \int_B u^{2c-2}F(\nabla u)^2 d\mu \\ \leq\ & -\lambda_1^2 \int_B u^{2c-2}F(\nabla u)^2 d\mu + 2(2c-3)\lambda_1 \int_B u^{2c-4}F(\nabla u)^4 d\mu \\ & +\frac{4}{3}\lambda_1 \int_B u^{2c-4}F(\nabla u)^4 d\mu \\ =\ & -\lambda_1^2 \int_B u^{2c-2}F(\nabla u)^2 d\mu + \left(4c-\frac{14}{3}\right)\lambda_1 \int_B u^{2c-4}F(\nabla u)^4 d\mu, \end{aligned} \tag{3.21}$$

consequently

$$\lambda_1 - \left(4c-\frac{14}{3}\right)I_c\lambda_1 + (2c-3)^2 I_c^2 \leq 0. \tag{3.22}$$

Considering $c > 2$, inequality (3.22) is solvable and answers satisfy in (3.5). □

4. Proof of the Main Results

Proof of Theorem 3.1. Let $\psi = u^c$, where

$$\psi|_{\partial B} = \frac{\partial \psi}{\partial \nu}|_{\partial B} = 0.$$

Then from Rayleigh-Ritz inequality, for any $c > 2$, we obtain

$$\begin{aligned} \Lambda_1 \ &\leq\ \frac{\int_B (\Delta\psi)^2 d\mu}{\int_B F(\nabla\psi)^2 d\mu} \\ &=\ \int_B [u^{2c-2}(\Delta u)^2 + 2(c-1)u^{2c-3}F(\nabla u)^2\Delta u \\ &\quad +(c-1)^2 u^{2c-4}F(\nabla u)^4]d\mu / \int_B u^{2c-2}F(\nabla u)^2 d\mu \\ &\leq\ \lambda_1 + (c-1)^2 I_c. \end{aligned} \tag{4.1}$$

By (3.5), we infer

$$\begin{aligned} \Lambda_1 \ &\leq\ \lambda_1 + (c-1)^2 I_c \\ &\leq\ \left[1 + \frac{(c-1)^2}{(2c-\frac{7}{3}) - \frac{2}{3}\sqrt{2(3c-4)}}\right]\lambda_1 \\ &=\ \left[1 + \frac{1}{3}\left(\frac{(3c-4)+1}{\sqrt{2(3c-4)}-1}\right)^2\right]\lambda_1. \end{aligned} \tag{4.2}$$

Taking $c = \dfrac{6+\sqrt{3}}{3}$, and minimizing the function $f(c) = \dfrac{(3c-4)+1}{\sqrt{2(3c-4)}-1}$, (4.1) becomes

$$\Lambda_1 \leq \frac{7+2\sqrt{3}}{3}\lambda_1.$$

Then, we get the proof. □

Proof of Theorem 3.2. For the first eigenvalue of Clamped plate problem, we get

$$\begin{aligned}
\Gamma_1 &\leq \frac{\int_B (\Delta\psi)^2 d\mu}{\int_B \psi^2 d\mu} \\
&= c^2 \int_B [u^{2c-2}(\Delta u)^2 + 2(c-1)u^{2c-3}F(\nabla u)^2 \Delta u \\
&\quad +(c-1)^2 u^{2c-4} F(\nabla u)^4] d\mu / \int_B u^{2c} d\mu \\
&= c^2 \big(\lambda_1^2 - \frac{2(c-1)}{2c-1}\lambda_1^2 + \frac{(c-1)^2}{2c-1}\lambda_1 I_c\big) \\
&= \frac{c^2}{2c-1}[\lambda_1^2 + (c-1)^2 \lambda_1 I_c]. \qquad (4.3)
\end{aligned}$$

Applying (3.5) into (4.3), we have

$$\begin{aligned}
\Gamma_1 &\leq \frac{c^2}{2c-1}\left[1 + \frac{(c-1)^2}{(2c-\frac{7}{3}) - \frac{2}{3}\sqrt{2(3c-4)}}\right]\lambda_1^2 \\
&= \sigma(c)\lambda_1^2, \qquad (4.4)
\end{aligned}$$

with

$$\sigma(c) = \frac{c^2}{2c-1}\left[1 + \frac{1}{3}\left(\frac{(3c-4)+1}{\sqrt{2(3c-4)}-1}\right)^2\right].$$

Minimizing this function based on the condition (3.4), completes the proof. □

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